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Absolute convergence ratio test

(1)

We can handle positive and alternating series but what about series with arbitrary signed terms?

Video example $\sum_{n=1}^{\infty} \frac{\sin n}{n^3} = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \dots$ (see Maple)

+ + + - - -

What to do?

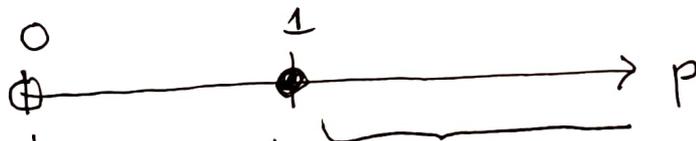
$\sum_{n=1}^{\infty} a_n \rightarrow \sum_{n=1}^{\infty} |a_n|$ is a positive series, test that, then see if signs matter for original series

example $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $0 < p \leq 1$ { diverges by the integral test
converges for $p > 1$

but $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ converges for $p > 0$ by alt. series test

so for $0 < p \leq 1$ this converges because of the alternating sign
The convergence is conditional on having the cancellations made by the alternating sign; while the corresponding absolute value series diverges.

$\sum_{n=1}^{\infty} a_n$ { is "conditionally convergent" if it converges but $\sum_{n=1}^{\infty} |a_n|$ diverges
is "absolutely convergent" if $\sum_{n=1}^{\infty} |a_n|$ converges



$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$:

conditionally convergent
(signs matter)

absolutely convergent
(signs don't matter)

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Absolute convergence implies convergence

Including signs can only make convergence easier but we need to prove this.

Assume $S = \sum_{n=1}^{\infty} |a_n|$ converges so $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$$\text{Then } 0 \leq a_n + |a_n| = \begin{cases} 0 & \text{if } a_n < 0 \\ 2a_n & \text{if } a_n > 0 \end{cases}$$

$$< 2|a_n|$$

$$0 < \sum_{n=1}^{\infty} (a_n + |a_n|) < \sum_{n=1}^{\infty} 2|a_n| = 2 \sum_{n=1}^{\infty} |a_n| \text{ converges}$$

must converge by comparison test

$$\underbrace{\sum_{n=1}^{\infty} (a_n + |a_n|)}_{\text{converges}} - \underbrace{\sum_{n=1}^{\infty} |a_n|}_{\text{converges}} = \sum_{n=1}^{\infty} ((a_n + |a_n|) - |a_n|) \text{ converges}$$

limit laws allow them to be combined if exist

$$= \sum_{n=1}^{\infty} a_n \text{ converges}$$

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video example $\sum_{n=1}^{\infty} \frac{\sin n}{n^3} \rightarrow \sum_{n=1}^{\infty} \frac{|\sin n|}{n^3} < \sum_{n=1}^{\infty} \frac{1}{n^3}$ convergent
p=3 series

so absolutely convergent,
hence convergent.

Maple: $\frac{1}{2} (\text{polylog}(3, e^{-i}) - \text{polylog}(3, e^i)) \approx 0.9429$

check out help page if curious.

Absolute convergence turns out to be of key importance for Taylor series. For positive series the comparison with a geometric series is a powerful tool. We can extend this to any series through absolute convergence.

$$\sum_{n=1}^{\infty} a r^{n-1} \rightarrow \sum_{n=1}^{\infty} |a| |r|^{n-1} \quad \left| \frac{a_{n+1}}{a_n} \right| = |r| < 1 \text{ for convergence}$$

original series absolutely convergent

compare to any series as $n \rightarrow \infty$

$$\sum_{n=1}^{\infty} a_n \rightarrow \sum_{n=1}^{\infty} |a_n| \rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \text{ if exists}$$

then looks more and more like a geometric series with ratio L

$$\left\{ \begin{array}{l} L < 1 \text{ converges} \\ L > 1 \text{ diverges terms growing!} \end{array} \right.$$

"absolute convergence ratio test"

$L = 1$ behaves like a p-series so inconclusive, need to compare directly to a p-series or use another test.

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example $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^3}{3^n}$ \leftarrow like p-series
 \leftarrow like geo series (beat p-series)!

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} = \left(\frac{n+1}{n} \right)^3 \frac{3^n}{3^n \cdot 3} = \frac{1}{3} \left(1 + \frac{1}{n} \right) \rightarrow \frac{1}{3} < 1$$

converges absolutely, so converges.

(notice p-series terms do not contribute to the limiting ratio)

example $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{(n+1)^p} \cdot n^p = \left(\frac{n}{n+1} \right)^p = \left(\frac{1}{1 + \frac{1}{n}} \right)^p \rightarrow 1 \text{ inconclusive.}$$

example $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ \leftarrow "hybrid"

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)(n+1)^n}{(n+1)n!} \frac{n!}{(n+1)n!}$$

$$= \left(\frac{n+1}{n} \right)^n = \left(1 + \frac{1}{n} \right)^n = e^1 = e > 1 \text{ diverges}$$

Fact $e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n} \right)^n$

$$= \lim_{n \rightarrow \infty} e^{\frac{\ln(1 + \frac{x}{n})}{1/n}} \quad \text{\% L'Hopital}$$

$$= e^{\lim_{n \rightarrow \infty} \frac{\ln(1 + x/n)}{1/n}}$$

$$= e^{\lim_{n \rightarrow \infty} \left(\frac{\frac{1}{1+x/n} \left(-\frac{x}{n^2} \right)}{-1/n^2} \right)} = e^{\lim_{n \rightarrow \infty} \frac{-x}{1+x/n}} \rightarrow 0$$

$$= e^x$$

Absolute convergence ratio test

Not so useful for us but let's understand it.
We can compare to a geometric series in another way.

$$\sum_{n=1}^{\infty} \underbrace{a r^{n-1}}_{a_n} \rightarrow |a_n|^{1/n} = (|a| |r|^{n-1})^{1/n}$$

$$= |a|^{1/n} |r|^{(n-1)/n}$$

$$= |a| \downarrow_0 |r| \uparrow_0 \rightarrow |a|^0 |r|^1$$

$$= |r| < 1$$

for convergence.

↓ apply to any series

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$$

$$|a_n|^{1/n} = \left(\frac{2^n}{n^n} \right)^{1/n} = \frac{2}{n} \rightarrow 0 \text{ as } n \rightarrow \infty$$

looks like it converges faster than any geo series since ratio goes to zero.

But we can also use the other comparison.

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{2^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n} = \frac{2 n^n}{(n+1)(n+1)^n} = \frac{2}{n+1} \left(\frac{n}{n+1} \right)^n$$

$$= \frac{2}{n+1} \left(1 - \frac{1}{n+1} \right)^n = \frac{2}{n+1} \frac{1}{e} \rightarrow 0 \text{ as } n \rightarrow \infty$$

A.C. Root test: $\lim |a_n|^{1/n} = L$

- ≤ 1 converges absolutely
- > 1 diverges
- $= 1$ inconclusive

but just a bit more convenient to use for some applications. not necessary since can always use ratio test.

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Hierarchy of factors in term expressions

fixed powers (polynomials)	n^p	$\frac{a_{n+1}}{a_n} \rightarrow 1$
fixed exponents	r^n	$\frac{a_{n+1}}{a_n} \rightarrow r$
factorials	$n!$	$\frac{a_{n+1}}{a_n} = n+1$
hybrid (power+exp)	$n^n = e^{n \ln n}$	$\frac{a_{n+1}}{a_n} \rightarrow e(n+1)$

upper rows "beaten" by lower rows