

## 11.5b Absolute and conditional convergence

①

$$\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n a_k \right)$$

$= S_n$  partial sum  $\rightarrow$  only "summable" for  
 1) geometric series (useful)  
 2) telescoping series (not so much)

Other series: test for convergence

1) positive series  $a_n > 0$

- integral test, comparison & limit comparison tests (error estimate)
- most useful comparisons: p-series, geometric series

2) alternating series:  $a_n = (-1)^n b_n$ ,  $b_n = |a_n|$  abs. value series

- alternating series test (error estimation)

3) no restriction on sign of  $a_n$ . What can we say?

Fact

$$|\underbrace{a + b}| \leq |\underbrace{a| + |b|}$$

useful upper bound

if opposite signs,  
cancellation makes  
this smaller

↓ extend to  $n$  terms

$$\left| \sum_{k=1}^n a_k \right| \leq \underbrace{\sum_{k=1}^n |a_k|}$$

if this converges  
as  $n \rightarrow \infty$

then modulo overall sign,  
this too much converge  
due to extra cancellation in sum

so it makes sense  
to compare a series  
to its absolute value  
series

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so we make the definitions:

$$\sum_{n=1}^{\infty} a_n \begin{cases} \text{is absolutely convergent if } \sum_{n=1}^{\infty} |a_n| \text{ converges} \\ \text{is conditionally convergent if } \sum_{n=1}^{\infty} |a_n| \text{ diverges} \\ \text{but } \sum_{n=1}^{\infty} a_n \text{ converges} \end{cases}$$

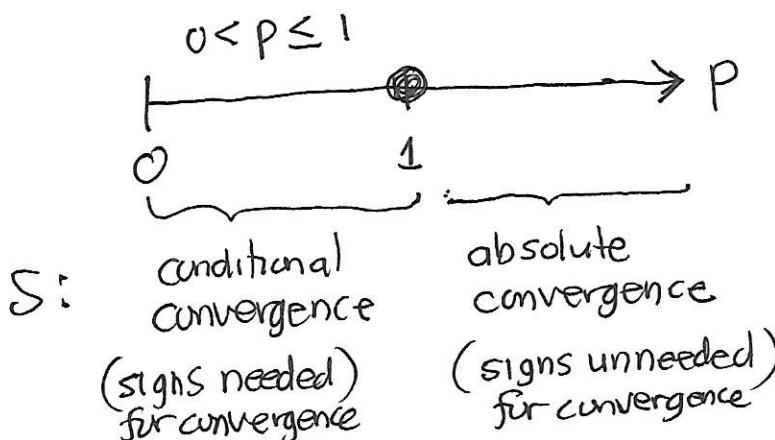
For abs. conv. the sign cancellation between terms is not needed for convergence.  
 For cond. conv. the signs are needed.

Alternating p-series are a perfect example.:

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} \quad \text{converges for } p > 0$$

The absolute value series is the p-series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{array}{l} \text{converges only for } p > 1 \\ \text{diverges for } 0 < p \leq 1 \end{array} \quad \text{so:}$$



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What good is absolute convergence?

Fact. If a series is absolutely convergent, it is convergent!

Proof.

$$0 \leq \underbrace{a_n + |a_n|}_{\begin{array}{l} = 0 \text{ if } a_n < 0 \\ = 2|a_n| \text{ if } a_n > 0 \end{array}} \leq 2|a_n|$$

Assume  $\sum_{n=1}^{\infty} |a_n|$  is convergent, so  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent.

$$\text{Then } \underline{a_n} = (\underbrace{a_n + |a_n|}_{1}) - (\underbrace{|a_n|}_{\text{cancel}})$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} [(a_{n+1} - a_n) + a_n]$$

limit of sum = sum of limits if both separate limits exist

$$\text{but } \sum_{n=1}^{\infty} (a_n + |a_n|) \leq \sum_{n=1}^{\infty} 2|a_n| = 2 \underbrace{\sum_{n=1}^{\infty} |a_n|}_{\text{converges}}$$

$$\text{so } \sum_{n=1}^{\infty} |a_n| = \left( \sum_{n=1}^{\infty} a_n + |a_n| \right) - \sum_{n=1}^{\infty} |a_n| \quad \begin{matrix} \text{converges} \\ \text{converges} \end{matrix}$$

so equal to previous line which must converge

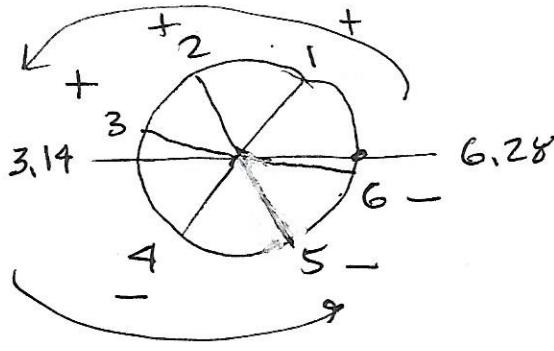
So for general series we can test the positive absolute value series to determine convergence.

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Video example (like example 8)

$$S = \sum_{n=1}^{\infty} \frac{\sin n}{n^3} \leftarrow \begin{array}{l} \text{can be positive or negative} \\ \text{but erratically} \end{array} \quad (\text{never zero})$$



integer radians  
on the unit  
circle

$\sin n > 0$  in  
upper half  
 $\sin n < 0$  in  
lower half

signs start out: + + + - - - + + + etc.  
(see Maple)

abs value series

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{n^3} \leftarrow \leq 1 \leq \sum_{n=1}^{\infty} \frac{1}{n^3} \quad p=3 \quad p\text{-series}$$

$p > 1$  convergent

so  $S$  is absolutely  
convergent  
so convergent.

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Revisit p-series with concrete values

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \rightarrow \sum_{n=1}^{\infty} \frac{1}{n^3}$$

convergent  $p=3$  series ( $p>1$ )  
 so abs. convergence  
 so convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1/3}} \rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$$

divergent  $p=1/3$  series ( $p \leq 1$ )  
 so diverges  
 but converges by  
 alternating series test  
 so conditionally convergent.