

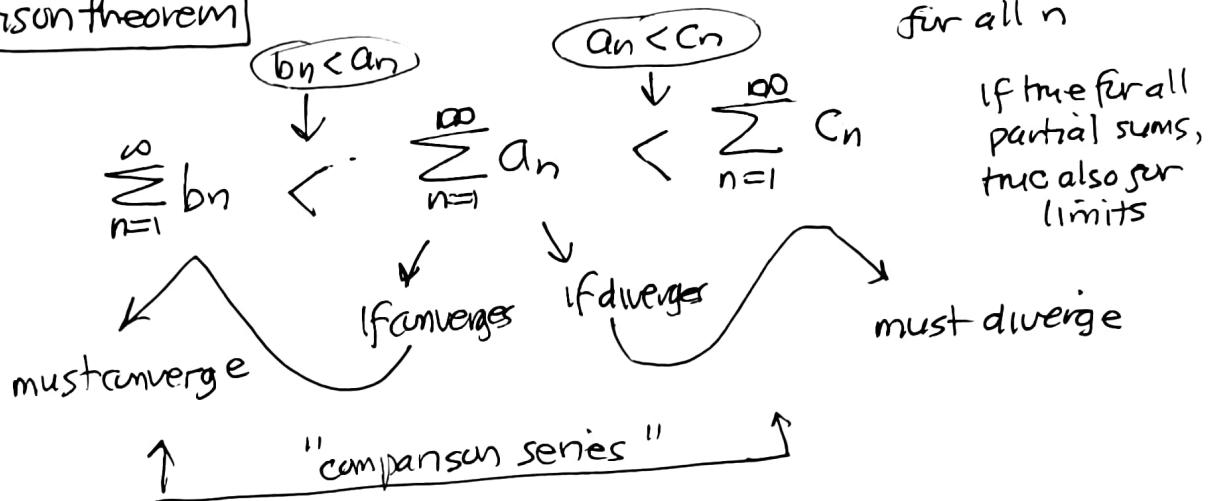
11.4

Positive series comparison tests

①

Last time we compared a positive infinite series with the area under a continuous function term by term to get obvious convergence/divergence properties. Now we compare directly to another simpler positive series whose convergence properties are known.

comparison theorem



But what can we compare to?

1) geometric series: $\sum_{n=1}^{\infty} ar^{n-1}$, $|r| < 1$ converges, otherwise diverges

2) p-series (or multiples): $\sum_{n=1}^{\infty} \frac{c}{n^p}$, $p > 1$ converges, otherwise diverges

Suppose we compare a p-series to a geometric series

$$\frac{a_{n+1}}{a_n} = r \text{ for a geometric series}$$

$$\text{p-series: } \frac{a_{n+1}}{a_n} = \frac{c}{(n+1)^p} = \frac{c}{n^p} \cdot \frac{n^p}{(n+1)^p} = \left(\frac{n}{n+1}\right)^p = \left(\frac{1}{1+\frac{1}{n}}\right)^p \rightarrow 1^-$$

$\downarrow 0 \text{ as } n \rightarrow \infty$

Conclusion: any geometric series converges faster than a p-series

11.4 Positive series comparison tests (2)

Video example $\sum_{n=1}^{\infty} \frac{5^n}{7^n+2} < \sum_{n=1}^{\infty} \frac{5^n}{7^n+0} = \sum_{n=1}^{\infty} \left(\frac{5}{7}\right)^n$ $r = \frac{5}{7} < 1$
 converges
 \downarrow
 converges

but $\sum_{n=1}^{\infty} \frac{5^n}{7^n-2}$
 $> \frac{5^n}{7^n}$
 \uparrow
 wrong direction
 useless

what to do? Only the behavior for $n \rightarrow \infty$ matters, but limiting behavior

$$\lim_{n \rightarrow \infty} \frac{5^n}{7^n-2} = 0 \text{ clearly but all convergent series must go to zero}$$

must compare limiting behavior:

$$\lim_{n \rightarrow \infty} \frac{5^n}{7^n-2} = \lim_{n \rightarrow \infty} \frac{5^n}{7^n} \cancel{-2} = \lim_{n \rightarrow \infty} \frac{1}{1-\frac{2}{7^n}} = 1$$

$\frac{5^n}{7^n}$ so have "same limiting behavior"
 so convergence properties should also be the same. Indeed this is true

Limit comparison theorem

$$\sum_{n=1}^{\infty} a_n \leftrightarrow \sum_{n=1}^{\infty} b_n, \quad \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1 \text{ means have same properties at large } n,$$

so either both converge or both diverge.

example $\sum_{n=1}^{\infty} \frac{7^n}{5^n+2}$
 $< \frac{7^n}{5^n}$ but useless
 $\uparrow r = \frac{7}{5} > 1$ divergent

$$\text{but } \lim_{n \rightarrow \infty} \frac{7^n}{5^n+2} = \lim_{n \rightarrow \infty} \left(\frac{7^n}{7^n} \frac{7^n}{5^n+2} \right) = \lim_{n \rightarrow \infty} \frac{7^n}{5^n+2} = \lim_{n \rightarrow \infty} \frac{1}{1+\frac{2}{7^n}} = 1$$

also diverges

Basically we can ignore or simplify an expression to one which has same limiting behavior in this sense, to determine convergence or divergence.

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Positive series comparison tests

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example

$$\sum_{n=1}^{\infty} \frac{2n^2+n+1}{3n^4+n+2}$$

$\sim \frac{2n^2}{3n^4} = \frac{2}{3n^2}$
convergent p-series
 $p=2 > 1$

$$\lim_{n \rightarrow \infty} \frac{2n^2+n+1}{3n^4+n+2} = \lim_{n \rightarrow \infty} \frac{3n^4 \cancel{2n^2+n+1}}{2n^4 \cancel{3n^4+n+2}}$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{2n^2}{3n^4} \cdot n^2(2+1/n+1/n^2)}{n^4(3+1/n^3+2/n^4)}$$

$$= \lim_{n \rightarrow \infty} \frac{3}{2} \left(\frac{2 + \cancel{1/n} + \cancel{1/n^2}}{3 + 1/n^3 + 2/n^4} \right) = \frac{1}{2}$$

obvious that this is the case, no real need to evaluate this limit.

$$\sum_{n=2}^{\infty} \frac{\ln n}{n}$$

can't simplify " this for large n , must use inequality
 $\ln n \geq 1$ for $n \geq 2$

$\rightarrow \sum_{n=2}^{\infty} \frac{1}{n}$ divergent $p=1$ harmonic series, so original diverges

example $\sum_{n=1}^{\infty} \frac{1}{n^n}$
↑ n in power (like geo series)
↑ n in base (like p-series)
 } "hybrid"

a) $\frac{1}{n^n} < \frac{1}{n^2}$ for $n \geq 2$, convergent $p=2$ series, so converges

or b) $\frac{1}{n^n} < \frac{1}{2^n}$ for $n \geq 2$, convergent $r = \frac{1}{2} < 1$ geo series, so converges

or c) $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^{n+1}}}{\frac{1}{n^n}} = \left(\frac{n}{n+1} \right)^n = \frac{1}{n+1} \left(\underbrace{\frac{n}{n+1}}_{\rightarrow e} \right)^n \underset{n \rightarrow \infty}{\cancel{\rightarrow}} \frac{1}{e(n+1)}$

ratio of successive terms gets smaller and smaller, approaching 0, smaller than any geometric series, so converges faster than any geometric series!

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Positive series comparison tests

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example

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$$

$$\frac{1}{n \cdot n^{\frac{1}{n}}}$$

$\sim \infty^0$ indeterminate \rightarrow
L'Hopital's rule

↓ compare

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges,}$$

so original diverges

$$n^{\frac{1}{n}} = (e^{\ln n})^{\frac{1}{n}} = e^{\frac{\ln n}{n}}$$

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{y_n}{1} = 0$$

$$\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = e^0 = 1$$

so indeed can ignore
original $\frac{1}{n}$ in formula.

example

$$\sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{2!} + \underbrace{\frac{1}{3!}}_{> 2^2} + \underbrace{\frac{1}{4!}}_{> 2^3} + \dots < 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

convergent geo series

but compare terms $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{1}{(n+1)} \xrightarrow{n \rightarrow \infty} 0$

converges faster than any geometric series.

1.4 Positive series comparison tests

(5)

[Estimating truncation error]

$$\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} b_n \leftarrow a_n < b_n \text{ strict inequality term by term.}$$

$$\sum_{n=1}^N a_n + R_N(a) \leftrightarrow \sum_{n=1}^N b_n + R_N(b)$$

↑ term by term in same inequality

↑ use estimates for comparison series to estimate error for original series.

example

$$\sum_{n=1}^{\infty} \frac{1}{n^3+1} < \frac{1}{n^3} \rightarrow \sum_{k=n+1}^{\infty} \frac{1}{n^3} < \int_n^{\infty} \frac{1}{x^3} dx = \left[\frac{x^{-2}}{-2} \right]_n^{\infty} = \frac{1}{2n^2}$$

truncate at n , error $< \frac{1}{2n^2}$ else

see Maple