

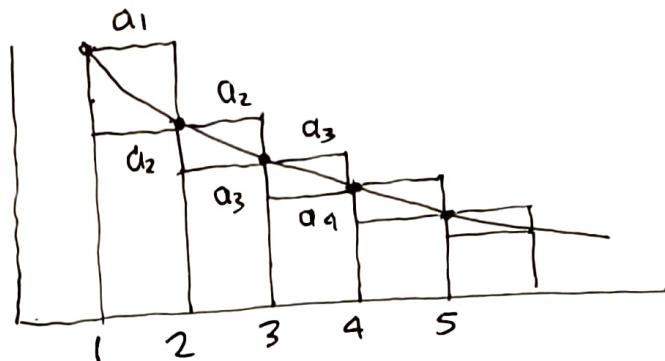
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### Integral test

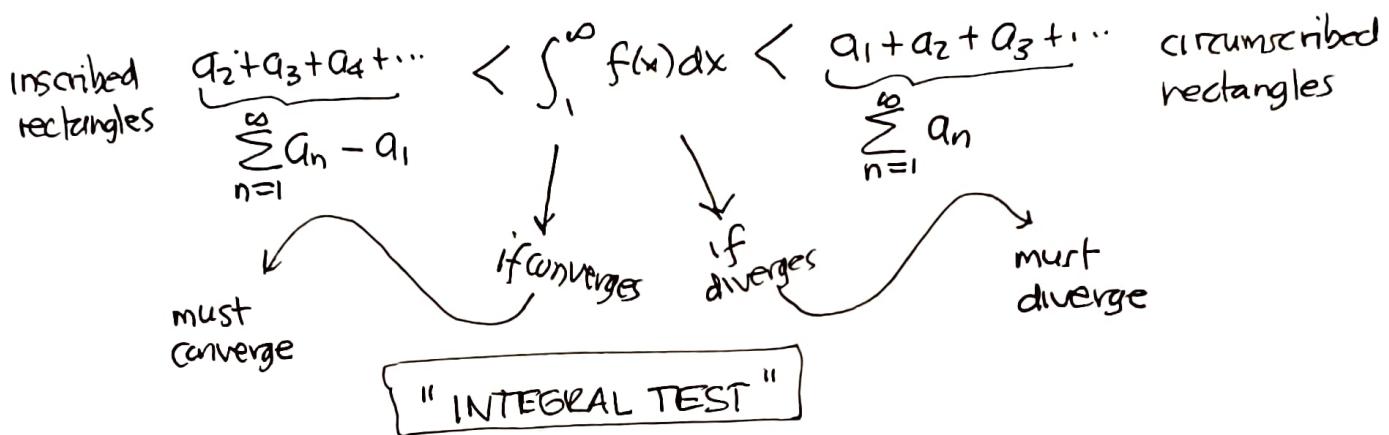
The close similarity between improper integrals and infinite series can be exploited to test series for convergence.

$$\textcircled{1} \quad \int_1^{\infty} f(x) dx, \quad f(x) > 0$$

$$\sum_{n=1}^{\infty} a_n, \quad a_n = f(n)$$



Assumes  $f(x)$  decreasing function  
 $f'(x) < 0$   
for  $x \geq 1$



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Integral test

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The easiest application of the integral test is to

p-series:  $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ,  $p > 0$  (so  $\lim_{n \rightarrow \infty} a_n = 0$ )

$$\text{Recall: } \int_1^{\infty} \frac{1}{x^p} dx = \int_1^{\infty} x^{-p} dx = \begin{cases} p \neq 1: \left| \frac{x^{1-p}}{1-p} \right|_1^{\infty} \text{ finite if } 1-p < 0 \\ p = 1: \left| \ln x \right|_1^{\infty} = \infty \end{cases} \quad p > 1$$

limit required!

converges for  $p > 1$

diverges for  $0 < p \leq 1$

same true for corresponding p-series

Examples:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{p-series with } p=2, \text{ converges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \quad \text{p-series with } p=1/2, \text{ diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{p-series with } p=1, \quad \text{"harmonic series"} \\ \text{diverges}$$

Fact  $\zeta(p) \equiv \sum_{n=1}^{\infty} \frac{1}{n^p}, \quad p > 1$  Riemann zeta function  
 defined complex  $n$   
 with  $\operatorname{Re}(n) > 1$ .

so Maple can evaluate these exactly.

It will turn out that p-series and geometric series are the only ones we need for analyzing convergence of Taylor series.

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other examples

(require us to be able to evaluate antiderivative)

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \rightarrow \int_1^{\infty} \frac{dx}{1+x^2} = \left[ \arctan x \right]_1^{\infty} = \lim_{t \rightarrow \infty} \arctan x \Big|_1^t$$

$$= \lim_{t \rightarrow \infty} (\arctan t - \arctan 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \approx 0.78$$

value unimportant

converges

converges



$$\text{Maple: } \cdot \frac{1}{2} (\pi \operatorname{coth}(\pi) - 1) \approx 1.077$$

$$< \underbrace{\frac{1}{1^2+1}}_{\text{a1}} + \int_1^{\infty} \frac{dx}{1+x^2} \approx 1.29$$

→  
due "bound" value  
but so what.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n} \rightarrow \int_1^{\infty} \frac{u \, du}{\cancel{x} \cancel{dx}} = \left[ \frac{1}{2} (\ln x)^2 \right]_1^{\infty} = \infty$$

diverges

$$\int u \, du = \frac{u^2}{2} + C$$

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2} \rightarrow \int_1^{\infty} \frac{\ln x \, dx}{x^2} = \begin{cases} x=\infty \\ x=1 \end{cases} \frac{u(e^u du)}{e^{2u}} = \int_{x=1}^{x=\infty} u e^{-u} du$$

$$\left[ \begin{array}{l} u = \ln x \rightarrow x = e^u \\ du = \frac{dx}{x} \quad dx = x \, du \\ = e^u du \end{array} \right]$$

$$= -(1+u) e^{-u} \Big|_{x=1}^{x=\infty}$$

$$= -(1+\ln x) \frac{1}{x} \Big|_{x=1}^{\infty}$$

$$= \lim_{t \rightarrow \infty} -\frac{(1+\ln x)}{x} \Big|_1^{\infty} = \lim_{t \rightarrow \infty} \left( -\frac{(1+\ln t)}{t} + 1 \right)$$

$\hookrightarrow \infty/\infty$

$$= \lim_{t \rightarrow \infty} \left( \underbrace{\frac{-1/t}{1}}_{=0} + 1 \right) = 1 \quad \text{converges}$$

Maple:  $-\zeta(1,2) \approx 0.938$  derivative of zeta function.

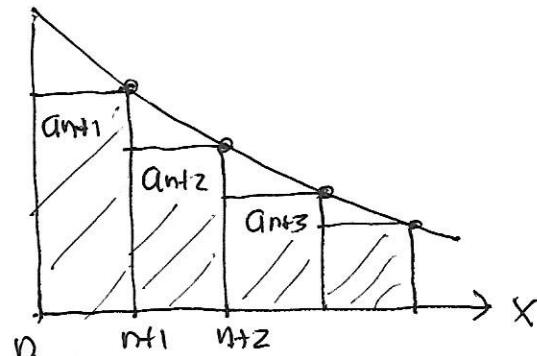
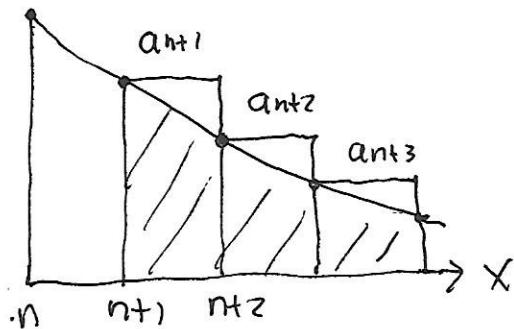
( $\ln n$  seems to act like  $n^0$  when multiplying  $1/n^p$ )

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Integral test

④

The integral test idea can also be used to estimate how many terms we need in a series to achieve a certain accuracy in our approximation to the infinite sum.

circumscribed  
rectangles:

$$\underbrace{\int_{n+1}^{\infty} f(x) dx}_{\equiv R_{n,L}} < \underbrace{\sum_{k=n+1}^{\infty} a_k}_{\equiv R_n} < \underbrace{\int_n^{\infty} f(x) dx}_{\equiv R_{n,R}}$$

remainder  
if truncate series :  $S_n = \sum_{k=1}^n a_k$       nth partial sum  
approximation

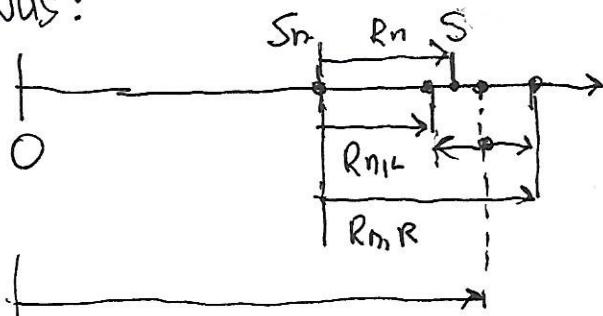
to  $S = \sum_{n=1}^{\infty} a_n$

$$S = S_n + R_n$$

$\hookrightarrow R_n = S - S_n$  but we can't  
evaluate the error in  
this approximation without  
knowing  $S$ .

must estimate it.

BONUS:



But we can narrow the  
uncertainty & improve  
the approximation

$$\underbrace{S_{n,better}}_{\text{corrected approximation}} = S_n + \frac{1}{2}(R_{n,L} + R_{n,R}) \quad \leftarrow \text{error less than } E_n = (R_{n,R} - R_{n,L}) / 2$$

much less than  $R_{n,K}$

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example  $p=3$  series

$$S = \sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.20206$$

$$\int x^{-3} dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} \quad (\text{comparison integral})$$

$$R_{n,R} = \int_n^{\infty} x^{-3} dx = \left[ -\frac{1}{2x^2} \right]_n^{\infty} = \frac{1}{2n^2} \quad \leftarrow \text{estimate for truncation error}$$

$$R_{n,R} = \int_{n+1}^{\infty} x^{-3} dx = \dots = \frac{1}{2(n+1)^2}$$

$$\frac{n=10}{S_{10}} = \sum_{n=1}^{10} \frac{1}{n^3} \approx 1.19753 \quad \text{error: } S - S_{10} \approx 0.0045$$

$$\text{estimate: } \frac{1}{2 \cdot 10^2} = 0.005 \quad \text{works!} \quad \uparrow$$

true error less than  
estimated max error

But corrected 10 term approximation?

$$\frac{1}{2}(R_{10,L} + R_{10,R}) = \frac{1}{2}\left(\frac{1}{2 \cdot 10^2} + \frac{1}{2 \cdot 11^2}\right) \approx 0.00457$$

$$S_{10,\text{better}} = S_{10} + \frac{1}{2}(R_{10,L} + R_{10,R}) \approx 1.202098$$

$$\text{new error } S - S_{10,\text{better}} \approx -0.000041 \quad \begin{matrix} 4 \text{ zeros.} \\ \wedge \end{matrix}$$

$$\text{predicted error max: } \frac{1}{2}(R_{10,R} - R_{10,L}) \approx \begin{matrix} 0.00043 \\ 3 \text{ zeros} \end{matrix} \quad \begin{matrix} \text{error bound} \\ (\text{abs value}) \end{matrix}$$

In other words we gained a factor of 10 in accuracy with the same # terms.

To get same result with truncation we would need  $n$  terms

$$\text{with } \frac{1}{2n^2} < 0.005 \rightarrow n > 31.6 \rightarrow 32 \text{ terms} \quad \begin{matrix} \text{compared to 10 terms} \end{matrix}$$

This is a nice example of how "numerical analysis" can improve approximations. Math minors take course on this topic.