$\boxed{11.11}$ Taylor series applications

Relativistic energy and momentum reduce to their nonrelativistic expressions as the speed of light c goes to infinity.

rest energy: $E_0 = mc^2$

energy: $E = \frac{mc^2}{\sqrt{1-v^2/c^2}} \rightarrow \infty$ as $v \rightarrow c$

kinetic energy: $K = E - E_0$

momentum: $p = \frac{mv}{\sqrt{1-v^2/c^2}} \rightarrow \infty$ as $v \rightarrow c$

$v < c$ for physical massive particles.

The binary series allows us to easily evaluate the nonrelativistic limit $v \ll c$ or $v/c \ll 1$. $\leftarrow (v = \text{speed})$

$$K = mc^2(1-v^2/c^2)^{-1/2} - mc^2 \qquad k = -1/2,\ x \rightarrow -v^2/c^2$$

$$= mc^2 \Big[\underbrace{(1 + \underbrace{(-v^2/c^2)}_{x})^{-1/2}} - 1 \Big]$$

$$\underbrace{\cancel{1} + \frac{(-\frac{1}{2})}{1} x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{1 \cdot 2} x^2 + \ldots \quad \cancel{-1}^{\,0}}$$

$$(-\tfrac{1}{2})(-\tfrac{v^2}{c^2}) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{1 \cdot 2}(-v^2/c^2)^2 + \ldots$$

(only first 2 terms needed to get NR result)

$$= mc^2 \Big[\tfrac{1}{2} v^2/c^2 + \tfrac{3}{8}(v^2/c^2)^2 + \ldots \Big]$$

$$= \frac{1}{2}mv^2 + \frac{3}{8}mv^2\left(\frac{v^2}{c^2}\right) + \ldots \qquad \text{(positive series, all signs cancel)}$$

$$= \underbrace{\tfrac{1}{2}mv^2}_{K_{NR}} \Big[1 + \underbrace{\tfrac{3}{4}\tfrac{v^2}{c^2} + \ldots}_{\text{fractional error}}\Big]$$

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numbers: typical NR speed $\quad v = 100\,m/s \approx 224\,mph$ (European fast trains!)

$c = 3 \times 10^8\ m/s$

$\frac{v}{c} = \frac{100}{3 \times 10^8} \approx 3 \times 10^{-7}$, $\left(\frac{v}{c}\right)^2 \sim 10 \times 10^{-14} \sim \underbrace{10^{-13}}$

$\sim$ fractional error

$$\cdot\ p = \frac{mv}{\sqrt{1-v^2/c^2}} = \frac{v}{c^2}E \approx \frac{v}{c^2}E_0$$

$$= mv(1-v^2/c^2)^{-1/2} = \underbrace{mv}_{p_{NR}}\Big(1 + \underbrace{\tfrac{1}{2}\tfrac{v^2}{c^2}}_{} + \ldots\Big)$$

fractional error $\sim 10^{-13}$

$\boxed{11.11}$ Taylor series applications

Black body radiation spectrum.

$$f(\lambda) = \frac{8\pi\hbar c\lambda^{-5}}{e^{\underbrace{\hbar c/\lambda kT}_{\ll 1}} - 1}$$

$\leftarrow$ longwavelength limit $\frac{1}{\lambda} \to 0^+$

$\ll 1$ in this limit, use e^x taylor series for $x \ll 1$

$$= \frac{8\pi\hbar c\,\lambda^{-5}}{\cancel{1} + \frac{\hbar c}{\lambda kT} + \frac{1}{2}\left(\frac{\hbar c}{\lambda kT}\right)^2 + \cdots - \cancel{1}}$$

$\to 0$ first term cancelled

$$= \frac{8\pi}{\lambda^5}\,\frac{\hbar c}{\left(\frac{\hbar c}{\lambda kT}\left[1 + \frac{1}{2}\left(\frac{\hbar c}{\lambda kT}\right) + \cdots\right]\right)}$$

$$= \frac{8\pi kT}{\lambda^4}\left(1 + \frac{1}{2}\frac{\hbar c}{\lambda kT} + \cdots\right)^{-1}$$

$$\approx \underbrace{\frac{8\pi kT}{\lambda^4}}_{\text{classical limit}}\left(1 - \underbrace{\frac{1}{2}\frac{\hbar c}{\lambda kT}}_{} + \cdots\right)$$

fractional correction for quantum effects

for $\frac{\lambda kT}{\hbar c} \gg 1$

$\lambda \gg \frac{\hbar c}{kT}$

only first 2 terms of binary series needed

correction comes from 3rd term