Taylor series : loose ends

once we have a Taylor series, we can manipulate it like we did for first geometric series & then binomial series.

example

$\int e^{-x^2}\,dx$ so let $x \to -x^2$ in the exponential power series & integrate term by term

no antiderivative formula!

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0} \frac{(-1)^n x^{2n}}{n!}$$

$$\int e^{-x^2}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{\int x^{2n}\,dx}{n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!\,(2n+1)} + C \quad \text{set} = 0$$

$$F(x) = \frac{x}{0!} - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots$$

$$= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots$$

$F(0) = 0 \leftarrow$ zero at $x = 0$ (no constant term)

$F(x) = \int_0^x e^{-t^2}\,dt$ = area accumulation function from $x = 0$.

so we have a Taylor series to evaluate normal probabilities

example: $\ln(1+3x^2)$?

$x \to 3x^2$ in $\ln(1+x)$, $|x|<1$

convergence: $|3x^2| < 1 \to |x| < 1/\sqrt{3}$ changes radius of convergence

Recall

$$\ln(1+x) = \int \frac{1}{1-(-x)}\,dx = \int \sum_{n=0}^{\infty} (-x)^n\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + \cancel{C}$$

$n=0$: $\frac{x^1}{1} = x \to 0$ → $\ln(1+0)=0$

$$\ln(1+3x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(3x^2)^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{3^{n+1} x^{2n+2}}{n+1} \quad 2n+2 \equiv 2m$$

reorder: $n+1 \equiv m \to n = m-1$

$n=0 \to m=1$

$$= \sum_{m=1}^{\infty} (-1)^{m-1} \frac{3^m x^{2m}}{m}$$

$$= \sum_{m=1}^{\infty} (-1)^{m-1} \frac{3^n x^{2n}}{n} = 3x^2 - \frac{9}{2}x^4 + \dots$$

can use Taylor series to evaluate limits as $x \to 0$:

$$\lim_{x\to 0} \frac{\ln(1+3x^2)}{\sin(2x^2)} = \lim_{x\to 0} \frac{3x^2 - \frac{9}{2}x^4 + \dots}{2x^2 - \frac{(2x^2)^3}{6} + \dots} = \lim_{x\to 0} \frac{x^2(3 - \frac{9}{2}x^2 + \dots)}{x^2(2 - \frac{8}{6}x^4 + \dots)}$$

$\sim \frac{0}{0}$

$\sin x = x - \frac{x^3}{3!} + \dots$

$$= \lim_{x\to 0} \frac{(3 - \frac{9}{2}x^2 + \dots)}{(2 - \frac{4}{3}x^4 + \dots)} = \frac{3}{2}$$

rest of series goes to zero

both series start at quadratic terms
so in this way only those "leading nonzero terms"
count in the limit as $x \to 0$.

see HW.

11.10c Taylor series: loose ends

Taylor remainder estimate

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}, \quad |x-a| < R \text{ converges}$$

$$= \underbrace{\sum_{n=0}^{N} \frac{f^{(n)}(a)(x-a)^n}{n!}}_{T_n(x)} + \underbrace{\sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}}_{R_n(x)}$$

$T_n(x)$: Taylor polynomial of degree n used to approximate $f(x)$

$R_n(x)$ = remainder after truncation of series

To prove that $f(x)$ equals the limit of its Taylor polynomial approximations, (where it converges) we need to show

$$\lim_{n\to\infty} R_n(x) = 0$$

Then

$$\underbrace{\lim_{n\to\infty} T_n(x)}_{\text{we know this converges}} = \lim_{n\to\infty} \left(f(x) - R_n(x)\right) = f(x) - \lim_{n\to\infty} R_n(x) = \underbrace{f(x)}$$

but now we know it converges to $f(x)$ if the remainder goes to zero

so we need a way to estimate $R_n(x)$ to force it to go to zero at large n.

Taylor remainder formula:

$$|R_n(x)| \le \frac{M}{(n+1)!} |x-a|^{n+1} \quad \leftarrow \quad \frac{f^{(n+1)}(a)(x-a)^{n+1}}{(n+1)!}$$

compared next term after nth term

↑ replace $f^{(n+1)}(a)$ by largest value of $f^{(n+1)}(x)$ on interval between x and a (for all n) (see Maple)

(technical) not appropriate for this level course