

II.10b

Binomial series

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From the geometric series formula:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1$$

$$\text{or } \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

we were able to manipulate with algebra, integration and differentiation to get power series representations of many other related functions.

We can generalize these same tricks.

$$\frac{1}{1+x} = (1+x)^{-1}$$

↓
let this exponent be any real number
(for $1+x > 0$)

$$f(x) = (1+x)^k = \sum_{n=0}^{\infty} c_n x^n ?$$

"binomial" = sum/diff of 2 terms

"binomial expansion"

$k > 0$ integer: this is a polynomial like

$$(1+x)^3 = \underbrace{(1+x)(1+x)(1+x)}_{\text{"expand" or multiply out}} = 1 + 3x + 3x^2 + x^3$$

↑ ↑ ↑ "binomial coefficients"

otherwise we get an infinite series

we can calculate it with the Taylor formula.

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Calculate:

$$f(x) = (1+x)^k$$

$$f'(x) = k(1+x)^{k-1}$$

$$f''(x) = k(k-1)(1+x)^{k-2}$$

$$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$$

$$f(0) = 1$$

$$f'(0) = k$$

$$f''(0) = k(k-1)$$

$$f'''(0) = k(k-1)(k-2)$$

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less than n

...

$$f^{(n)}(0) = \underbrace{k(k-1)(k-2)\dots}_{\text{n factors}} \underbrace{(k-(n-1))}_{=k-n+1}$$

$$\frac{f^{(n)}(0)}{n!} = \frac{k(k-1)(k-2)\dots(k-n+1)}{1 \cdot 2 \cdot 3 \cdots n} \quad \begin{matrix} \leftarrow \\ n \text{ factors top \& bottom, } n \geq 1 \end{matrix}$$

$$(1+x)^k = 1 + \sum_{n=1}^{\infty} \frac{k(k-1)(k-2)\dots(k-n+1)}{1 \cdot 2 \cdot 3 \cdots n} x^n$$

convergence?

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{k(k-1)(k-2)\dots(k-n+1)(k-n)}{1 \cdot 2 \cdot 3 \cdots n \cdot (n+1)} |x|^{n+1}}{\frac{k(k-1)(k-2)\dots(k-n+1)}{1 \cdot 2 \cdot 3 \cdots n} |x|^n} = \frac{|k-n|}{n+1} |x|$$

$$= \frac{(|k-n|/n)}{(n+1)/n} |x| = \frac{|\frac{k}{n}-1|}{1+\frac{1}{n}} |x| \xrightarrow{n \rightarrow \infty} \frac{0-1}{1+0} |x| = |x| < 1$$

radius of convergence ↑

same as for geometric series: $k = -1$

$$(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n, |x| < 1$$

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example

$$f(x) = \frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}}$$

$z=2 \cdot 2 \cdot 1$

$$= 1 + \frac{(-\frac{1}{2})}{1} x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{1 \cdot 2} x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1 \cdot 2 \cdot 3} x^3 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{1 \cdot 2 \cdot 3 \cdot 4} x^4 + \dots$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(1 \cdot 3 \cdot 5 \dots (2n-1))}{2^n n!} x^n$$

signs alternate since
one more negative factor
each time

example

$$\frac{1}{\sqrt{1-x^2}} = (1+(-x^2))^{-\frac{1}{2}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^n n!} (-x^2)^n$$

$x \rightarrow -x^2$:
 $| -x^2 | < 1$
so still $|x| < 1$
for convergence

$$= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^n n!} x^{2n}$$

next:

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C$$

integrate term by term:

$$\arcsin(x) = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^n n!} \frac{x^{2n+1}}{2n+1}$$

constant term 0
since $\arcsin(0) = 0$ but can be rewritten:

good enough
for
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$$n=3: \quad \frac{1 \cdot 3 \cdot 5}{2^3 3!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{2 \cdot 4 \cdot 6} \quad \frac{1}{(2^3 3!)^2} = \frac{(2 \cdot 3)!}{(2^3)^2 (3!)^2} = \frac{(2n)!}{2^{2n} (n!)^2}$$

$(2 \cdot 1)(2 \cdot 2)(2 \cdot 3)$
 $= 2^3 3!$

$$\arcsin(x) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (n!)^2} \frac{x^{2n+1}}{2n+1}$$

$= 4^n$

(works for $n=0$!)

check wiki!

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In practice no need to find general formula for n th term
if only lowest terms needed!

First order:

$$(1+x)^k = 1+kx \quad \text{very useful!}$$

$$\begin{aligned} \text{or: } \sqrt{1-x^2} &= (1+(-x^2))^{-1/2} \\ &= 1 - \frac{1}{2}(-x^2) - \frac{1}{2} \frac{(-\frac{1}{2})}{1 \cdot 2} (-x^2)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1 \cdot 2 \cdot 3} (-x^2)^3 + \dots \\ &= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \dots \end{aligned}$$

Changing radius of convergence from $|x| < 1$

$$\begin{aligned} \sqrt{4+x^2} &= \sqrt{4(1+\frac{x^2}{4})} = 2(1+(\frac{x^2}{4}))^{1/2} \\ &\quad x \rightarrow \frac{x^2}{4} \quad |\frac{x^2}{4}| < 1 \rightarrow |x| < 2 \text{ converges} \\ &= 2 \left[1 + \frac{1}{2}(\frac{x^2}{4}) + \frac{1}{2} \frac{(-\frac{1}{2})(\frac{x^2}{4})^2}{1 \cdot 2} + \frac{1}{2} \frac{(-\frac{1}{2})(-\frac{3}{2})(\frac{x^2}{4})^3}{1 \cdot 2 \cdot 3} + \dots \right] \\ &= 2 \left[1 + \frac{1}{8}x^2 - \frac{1}{8 \cdot 16}x^4 + \frac{1}{16 \cdot 48}x^6 - \dots \right] \end{aligned}$$

or try to find formula for n th term (tedious)
(Maple cannot do this for us!)

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$$\sqrt[p]{(1-x)} = (1-x)^{\frac{1}{p}} \quad p = 2, 3, 4, 5 \text{ etc.}$$

$$= (1+(-x))^{\frac{1}{p}} = 1 + \frac{1}{p}(-x) + \frac{1}{p} \frac{(\frac{1}{p}-1)(-\frac{1}{p})}{1 \cdot 2} (-x)^2 + \dots$$

example: square root

$$\sqrt{1-x} = (1+(-x))^{1/2} = 1 + \frac{1}{2}(-x) + \frac{1}{2} \frac{(-\frac{1}{2})(-\frac{3}{2})}{1 \cdot 2} (-x)^3$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$$

→ all negative after $n=0$.

look first → then overall negative sign.
then p^n in denominator

only one choice, but try to derive that formula

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continue:

$$(1-x)^{1/2} = 1 - \frac{x}{2} + \underbrace{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}_{(n=2)} \cancel{(x)^2} + \underbrace{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}_{(n=3)} \cancel{(x)^3} + \underbrace{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}_{(n=4)} \cancel{(x)^4}$$

overall minus sign

$2^n n!$ in denominator

numerator factors

$$\begin{matrix} 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \end{matrix} \quad \left. \begin{matrix} \text{odd integers minus 1} \\ \text{beginning with 1 when } n=2 \end{matrix} \right\}$$

odd integers minus 1 but beginning with 1 when $n=2$

last integer factor is

$$2(n-1) - 1 \quad = 1 \text{ when } n=2$$

$\left. \begin{matrix} \text{---} \\ = 1 \text{ when } n=2 \\ \text{---} \end{matrix} \right.$

$$= 2n-3$$

Conclude:

$$(1-x)^{1/2} = 1 - \frac{x}{2} - \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \dots (2n-3)}{2^n n!} x^n$$

How does it change for other roots?

$$(1-x)^{1/3} = 1 - \frac{x}{3} - \sum_{n=2}^{\infty} \frac{\text{numer}}{3^n n!} x^n$$

R-factors: $\frac{1}{3} \left(-\frac{2}{3} \right) \left(-\frac{5}{3} \right) \left(-\frac{8}{3} \right)$ numer: 2, 5, 8, 11 multiples of 3 minus 1

start: $3(2-1)-1 = 2$ when $n=2$

$3(n-1)-1 = 3n-4$ for general n .

This is too tedious to expect you to come up with these formulas, so use process of elimination among formulas given to you.

overall minus, denominator obvious, formula should match third term in series.
you have 5 tries! (to start).

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Last trick:

$$f(x) = \frac{x^3}{\sqrt{4+x^2}} = \frac{x^3}{\sqrt{4(1+\frac{x^2}{4})}} = \frac{x^3}{2} \left(1 + \underbrace{\left(\frac{x^2}{4}\right)}_{x \rightarrow \frac{x^2}{4}}\right)^{-1/2}$$

$\left|\frac{x^2}{4}\right| < 1 \rightarrow |x| < 2$

multiply binomial
 series by $\frac{x^3}{2}$
 combine powers of x