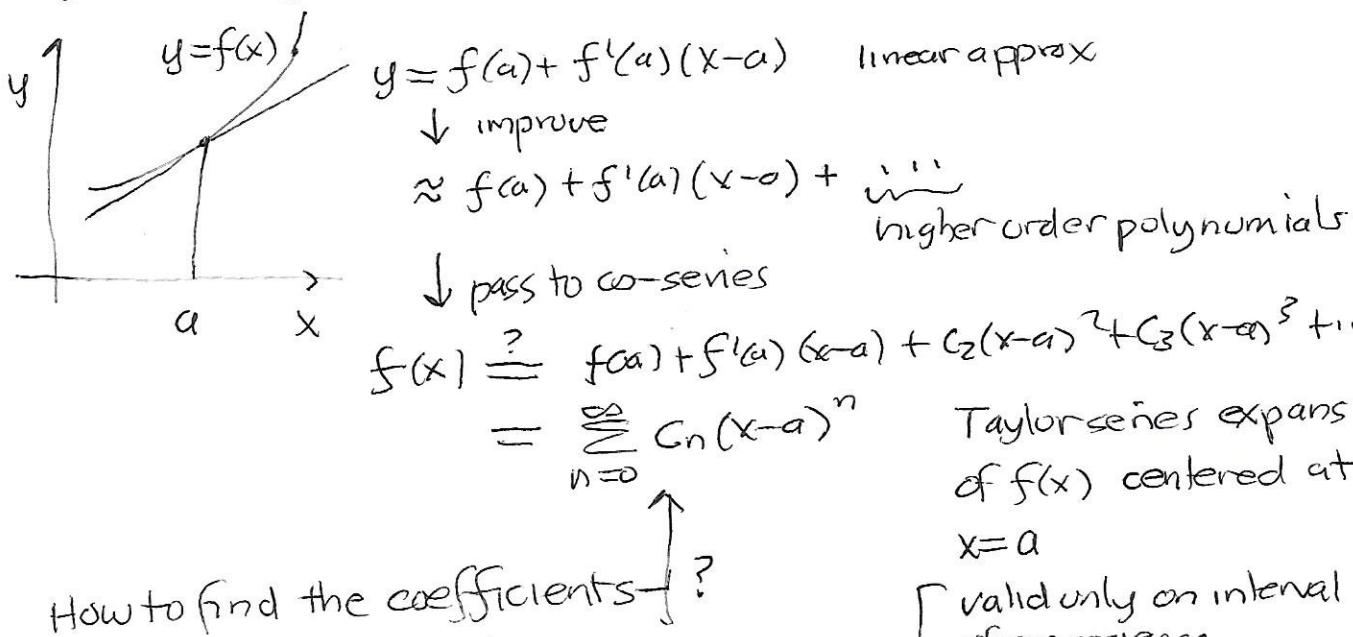


11.10

## Taylor series

(1)

After 50 years of using Taylor series I still can't remember the term MacLaurin series — you only find it in calc textbooks! Forget it. They are all Taylor series.



We need to match not only  $f(a)$  and  $f'(a)$  as in the linear approx, but all derivatives at  $x=a$ .

$$\text{Let } F(x) = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

$$F(x) = f(a) + f'(a)(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots \quad F(a) = f(a)$$

$$F'(x) = f'(a) + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots \quad F'(a) = f'(a)$$

$$F''(x) = 2c_2 + 3 \cdot 2c_3(x-a) + 4 \cdot 3 \cdot 2c_4(x-a)^2 + \dots \quad F''(a) = 2c_2$$

$$F^{(3)}(x) = 3 \cdot 2 \cdot c_3 + 4 \cdot 3 \cdot 2c_4(x-a) + \dots \quad F^{(3)}(a) = 3 \cdot 2c_3$$

$$F^{(4)}(x) = 4 \cdot 3 \cdot 2 \cdot c_4 + \dots \quad F^{(4)}(a) = 4 \cdot 3 \cdot 2c_4$$

$$F^{(n)}(x) = \dots$$

Conclusion set  $F^{(n)}(a) = n! c_n = \frac{f^{(n)}(a)}{n!}$

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$$n > 0: F^{(n)}(a) = n! c_n$$

$n=0: \text{define } 0! = 1$
$\overline{f^{(0)}(a) = f(a)}$
$c_0 = \frac{f^{(0)}(a)}{0!} = f(a)$

$$F(x) = \left[ \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \right] \stackrel{?}{=} f(x), \quad |x-a| < R$$

We can check where this converges BUT we still have to show that  $F(x) = f(x)$  which requires estimating the error. We will do this on day 3.

## 11.10 Taylor series

(2)

Warning!

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

Not so fast! Being able to evaluate  $f^{(n)}(a)$  as an explicit function of  $n$  is similar to being able to get a formula for the sum of a finite number of terms (partial sums) as an explicit function of  $n$  so that we can take the limit to evaluate exactly the infinite sum

$$S_n = \sum_{k=1}^n a_k \rightarrow S = \sum_{n=0}^{\infty} a_n = \lim_{n \rightarrow \infty} S_n$$

need to "sum"  
this to a formula in  $n$

in order to take its limit as a function of  $n$ .

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)(x-a)^k}{k!} = \text{nth Taylor polynomial}$$

We truncate the series to approximate the function  $f(x)$  but we then need to estimate the error to know how many terms we need to achieve a given precision.

II, 10)

Taylor series

(3)

Easiest case

$$f(x) = e^x$$

$$f'(x) = e^x$$

$$f^{(n)}(x) = e^x \rightarrow f^{(n)}(0) = e^0 = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{|x|^{n+1}}{(n+1)!} / \frac{|x|^n}{n!} = \frac{|x| n!}{(n+1)!} = \frac{|x|}{n+1} \rightarrow 0 \text{ as } n \rightarrow \infty$$

so this converges for all values of  $x$ ,

Tricks with  $e^x$ 

$$\begin{aligned} \text{video example } X e^{-2x} &= X \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{n!} x^{n+1} \\ &= \sum_{m=1}^{\infty} (-1)^{m-1} \frac{2^{m-1} x^m}{(m-1)!} \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{n-1} x^n}{(n-1)!} \end{aligned}$$

$m = n+1$   
 $n = m-1$   
 $m = 1 \text{ when } n=0$

forall  $x$  converges  
(powerseries tricks  
dont change convergence  
everywhere)

expansion at  $x=2$ ?

no need to re derive:

$$e^x = e^{x-2+2} = e^{x-2} e^2 = e^2 \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^2 (x-2)^n}{n!}$$

hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \right) = \sum_{n=0}^{\infty} \underbrace{\frac{1}{2} [1 + (-1)^n]}_{\begin{cases} 0, n \text{ odd} \\ 1, n \text{ even} \end{cases}} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad \text{only even terms from } e^x!$$

$$\sinh x = \frac{e^x - e^{-x}}{2} = \dots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \text{only odd terms from } e^x!$$

$$= \sum_{n=0}^{\infty} \underbrace{\frac{1}{2} [1 - (-1)^n]}_{\begin{cases} 1, n \text{ odd} \\ 0, n \text{ even} \end{cases}} \frac{x^n}{n!}$$

$$= \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

(4)

## 11.10 Taylor Series

Recall we found a power series for  $\ln(1+x)$  using geometric series tricks. We can actually derive it directly from the Taylor formula.

$$f(x) = \ln(1+x) \longrightarrow$$

$$f(0) = \ln 1 = 0$$

$$f'(x) = \frac{1}{1+x} = (x+1)^{-1}$$

(so series starts with  $n=1$ )

$$f''(x) = (-1)(x+1)^{-2}$$

$$f^{(3)}(x) = (-1)(-2)(x+1)^{-3}$$

$$f^{(4)}(x) = (-1)(-2)(-3)(x+1)^{-4}$$

$$n>0: f^{(n)}(x) = \underbrace{(-1)(-2)\dots(-(n-1))}_{(-1)^{n-1}(n-1)!} (x+1)^{-n} \longrightarrow f^{(n)}(0) = (-1)^{n-1}(n-1)!$$

$$\frac{f^{(n)}(0)}{n!} = (-1)^{n-1} \frac{(n-1)!}{n!}$$

$$= (-1)^{n-1} \frac{(n-1)!}{n \cdot (n-1)!} = (-1)^{n-1} \frac{1}{n}$$

so:

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

rare that  
one can find  
a simple formula  
for  $f^{(n)}(a)$

convergence?

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{\frac{n+1}{n}} = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right) |x|^n$$

$$= \underbrace{\lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)}_{=1} |x| = |x| \stackrel{\text{as } \frac{1}{n} \rightarrow 0}{\text{for convergence.}}$$

(as for geometric series)

endpoints:  $x = \pm 1$

$$x=1: \ln 2 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

alternating harmonic series (converges) (but very slowly!)

$$x=-1: \ln(0) = \sum_{n=0}^{\infty} (-1)^n \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^n = -\underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\rightarrow \infty} = -\infty$$

harmonic series  
diverges

(must diverge!)

11.10	<u>Taylor series</u>	(5)
After $e^x$ and $\ln x$ ( $\rightarrow \ln(x+1)$ ), the trig functions are next!		
$f(x) = \sin x$	$f(0) = 0 \rightarrow f^{(0)}(0) = 0 \rightarrow \text{even } 0$	
$f'(x) = \cos x$	$f'(0) = 1 \rightarrow f^{(1)}(0) = 1 \rightarrow \text{odd } +1$	
$f''(x) = -\sin x$	$f''(0) = 0 \rightarrow f^{(2)}(0) = 0 \rightarrow \text{even } 0$	
$f'''(x) = -\cos x$	$f'''(0) = -1 \rightarrow f^{(3)}(0) = -1 \rightarrow \text{even } -1$	
$f^{(4)}(x) = \sin x$	$\underbrace{\hspace{10em}}$ repeats every 4 terms	alternating sign odd powers $n$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

odd terms from  $e^x$   
with extra alternating sign  
(valid for all  $x$ )

tricks!  $\frac{d}{dx}$  ↓

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

even terms from  $e^x$   
with extra alternating sign

at  $x = \pi/3 \neq 0$ :

$$\begin{aligned} \sin x &= \sin\left(x - \frac{\pi}{3} + \frac{\pi}{3}\right) = \sin\left(x - \frac{\pi}{3}\right) \underbrace{\cos \frac{\pi}{3}}_{1/2} + \cos\left(x - \frac{\pi}{3}\right) \underbrace{\sin \frac{\pi}{3}}_{\sqrt{3}/2} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{\left(x - \frac{\pi}{3}\right)^{2n+1}}{(2n+1)!} + \frac{\sqrt{3}}{2} \sum_{n=0}^{\infty} (-1)^n \frac{\left(x - \frac{\pi}{3}\right)^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2} \left[ \frac{\left(x - \frac{\pi}{3}\right)^{2n+1}}{(2n+1)!} + \frac{\left(x - \frac{\pi}{3}\right)^{2n}}{(2n)!} \right] \end{aligned}$$

$$f(x) = x \sin x \Rightarrow x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)!}$$

$$= \sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^{2m}}{(2m-1)!}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{(2n-1)!}$$

$$\begin{aligned} 2n+2 &= 2m \\ n+1 &= m \\ n &= m-1 \\ m &= 1 \text{ when } n=0 \\ 2n+1 &= 2m-1 \end{aligned}$$

(11.10)

Taylor series

④

$$\text{Recall } \arctan(x) = \int_0^x \frac{1}{1+t^2} dt = \sum_{n=0}^{\infty} \int_0^x (-t^2)^n dt = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1} \Big|_0^x$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \dots \text{ easy but}$$

suppose we try to evaluate directly from the Taylor formula.

$$f(x) = \arctan x$$

$$f'(x) = (1+x^2)^{-1}$$

$$f''(x) = -(1+x^2)^{-2}(2x)$$

$$f'''(x) = (-1)(-2)(1+x^2)^{-3}(2x)(2x) - (1+x^2)^{-2}(2)$$

$$f^{(4)}(x) = (-1)(-2)(-3)(1+x^2)^{-4}(2x)(2x)(2x) + (-2)(1+x^2)^{-3}(2x)(2) \quad f^{(4)}(0) = 0$$

$$+ (-1)(-2)(1+x^2)^{-3} 4(2x) - (-2)(1+x^2)^{-3}(2x)(2) \quad f^{(5)}(0) = 16 + 8$$

$$f^{(5)}(x) = \dots \text{ (all with factor of } x \text{)} + (-1)(-2)(1+x^2)^{-3}(4) \quad = 24$$

$$(-1)(-2)(1+x^2)^{-3} \overbrace{(8)}$$

$$24 = 4 \cdot 3 \cdot 2 = 4!$$

$$\arctan x = \frac{1}{1}x - \frac{2}{3!}x^3 + \frac{24}{5!}x^5 - \dots$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots$$

?? difficult to "guess" pattern  
even if we use Maple to do the  
evaluation (see Maple worksheet)

Conclusion? Tricks matter in practice.