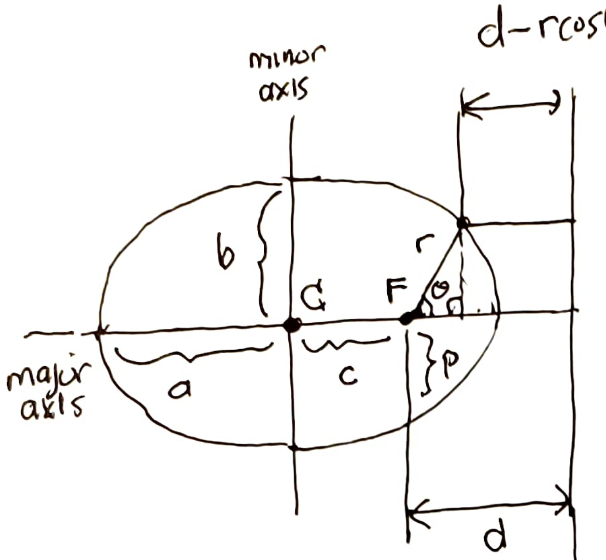
10.5 Fun with ellipses (and review for rest of chapter)



directrix (vertical line)

Center C
Focus F ≐ origin polar coords

Cartesian coords:
$x = r\cos\theta,\ y = r\sin\theta$

$$\underbrace{r}_{\text{radial distance}} = \underbrace{e}_{\text{proper fractional eccentricity}} \underbrace{(d - r\cos\theta)}_{\text{distance from directrix}}$$

$\boxed{\text{solve for } r!}$

$$r = ed - re\cos\theta$$
$$ed = r(1 + e\cos\theta)$$
$$\boxed{r = \frac{ed}{1+e\cos\theta} \equiv \frac{p}{1+e\cos\theta}}$$

polar curve form.

convert to Cartesian coords:

$$\underbrace{r^2}_{x^2+y^2} = e^2(d - \underbrace{r\cos\theta}_{x})^2 = e^2(d^2 - 2dx + x^2) = e^2x^2 - 2de^2x + e^2d^2$$

$$x^2(1-e^2) + 2de^2x + y^2 = e^2d^2$$

$$x^2 + \frac{2de^2}{1-e^2}x + \frac{y^2}{1-e^2} = \frac{e^2d^2}{1-e^2}$$

$$\Big(x + \underbrace{\frac{de^2}{1-e^2}}_{\equiv -h}\Big)^2 - \Big(\frac{de^2}{1-e^2}\Big)^2 + \frac{y^2}{1-e^2} = \frac{e^2d^2}{1-e^2}$$

$$(x-h)^2 + \frac{y^2}{1-e^2} = \frac{e^2d^2}{1-e^2} + \frac{d^2e^4}{(1-e^2)^2} = \frac{e^2d^2(1\cancel{-e^2}\cancel{+e^2})}{(1-e^2)^2} = \frac{e^2d^2}{(1-e^2)^2} \equiv a^2$$

$$\frac{(x-h)^2}{a^2} + \frac{y^2}{\underbrace{a^2(1-e^2)}_{\equiv b^2}} = 1$$

center is $|h|$ units left of focus

$$\boxed{\frac{(x-h)^2}{a^2} + \frac{y^2}{b^2} = 1}$$

$$\frac{b^2}{a^2} = 1 - e^2 \rightarrow e^2 = 1 - \frac{b^2}{a^2} \Rightarrow \boxed{e = \sqrt{1 - \frac{b^2}{a^2}}}$$

$$\boxed{b = a\sqrt{1-e^2}}$$
$$\boxed{a = \frac{p}{1-e^2}}$$
$$\boxed{p = a(1-e^2)}$$

$\boxed{10.5}$ Fun with ellipses

To make all this concrete, let's pick a pleasingly shaped ellipse

$$\frac{x^2}{3^2}+\frac{y^2}{2^2}=1 \qquad a=3,\ b=2,\ e=\sqrt{1-\frac{4}{9}}=\frac{\sqrt{5}}{3}$$

$$p=3\left(1-\frac{5}{9}\right)=3\cdot\frac{4}{9}=4/3$$

$$\downarrow$$

$$y=\pm 2\sqrt{1-\frac{x^2}{9}} \quad \longleftrightarrow \quad r=\frac{4/3}{1-\frac{\sqrt{5}}{3}\cos\theta}$$

same ellipse but shifted horizontally

$$\begin{cases} x=3\cos t \\ y=2\sin t \\ t=0..2\pi \end{cases} \quad \nearrow$$

same area, same circumference

Set up and evaluate the six integrals (using Maple):

$$A=2\int_{-3}^{3}|y(x)|\,dx = 2\int_0^{\pi} y(t)\underbrace{(-dx(t))}_{>0} = \int_0^{\pi}\frac{1}{2}r^2\,d\theta$$

$$C=2\int_{-3}^{3}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{2\pi}\sqrt{x'^2+y'^2}\,dt = \int_0^{2\pi}\sqrt{r^2+r'^2}\,d\theta$$