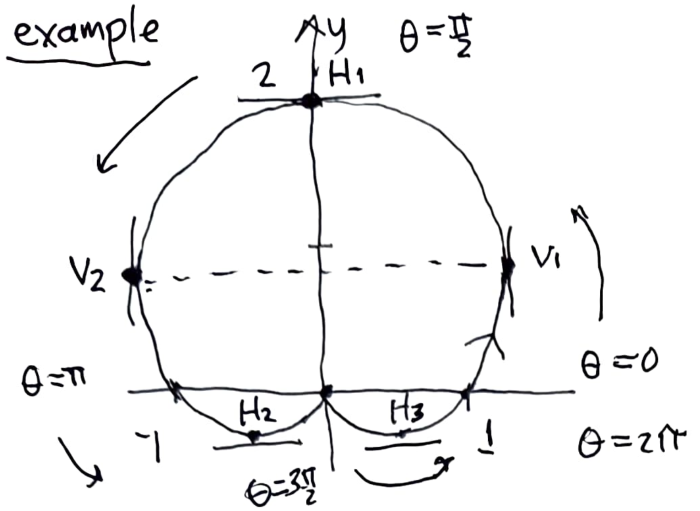
# $\boxed{10.4d}$ polar curve tangents

example




cardioid $r = 1+\sin\theta$, $\theta = 0..2\pi$

From the plot it is clear where the horizontal and vertical tangents occur so how do we find these special points?

More generally how do we get the slope of the tangent line at any point on the curve and then its equation?

A polar curve is a special case of a parametrized curve. Just go back to the Cartesian coordinates and use the chain rule:

$$r = r(\theta) \rightarrow \quad x = r(\theta)\cos\theta\ , \quad y = r(\theta)\sin\theta .$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \ldots$$

NO NEED FOR BOXED Formula, just do it.

$$x(\theta) = (1+\sin\theta)\cos\theta = \cos\theta + \sin\theta\cos\theta$$

$$y(\theta) = (1+\sin\theta)\sin\theta = \sin\theta + \sin^2\theta$$

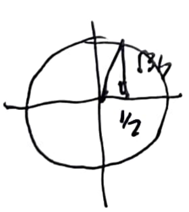
$$x'(\theta) = -\sin\theta + (\cos\theta)\cos\theta + \sin\theta(-\sin\theta) = -\sin\theta + \underbrace{\cos^2\theta - \sin^2\theta}_{1-\sin^2\theta}$$

$$= 1 - \sin\theta - 2\sin^2\theta \underset{\text{factor}}{=} (1+\sin\theta)(1-2\sin\theta)$$

$$y'(\theta) = (\cos\theta) + 2\sin\theta\cos\theta \underset{\text{factor}}{=} (1+2\sin\theta)\cos\theta$$

$$\frac{dy}{dx} = \frac{\cos\theta\ (1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

At $\theta = \frac{\pi}{3}$:




$\cos\frac{\pi}{3} = \frac{1}{2}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$r = 1 + \sin\frac{\pi}{3} = 1 + \frac{\sqrt{3}}{2}$

$$\frac{dy}{dx} = \frac{\frac{1}{2}\left(1 + 2\left(\frac{\sqrt{3}}{2}\right)\right)}{\left(1+\frac{\sqrt{3}}{2}\right)\left(1 - 2\left(\frac{\sqrt{3}}{2}\right)\right)} = \frac{1+\sqrt{3}}{\underbrace{(2+\sqrt{3})(1-\sqrt{3})}_{=2-\sqrt{3}-3 \,=\, -(1+\sqrt{3})}}$$

$$= -1$$

$\boxed{10.4\ d}$ polar curve tangents



tangent line through pt:

$$x = \left(1+\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}(2+\sqrt{3}) = x_0$$

$$y = \left(1+\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{4}(2\sqrt{3}+3) = y_0$$

$$y = y_0 + m(x-x_0) = \frac{1}{4}(2\sqrt{3}+3) - 1\left(x - \frac{1}{4}(2+\sqrt{3})\right)$$

$$= \frac{1}{4}(2\sqrt{3}+3) + \frac{1}{4}(2+\sqrt{3}) - x$$

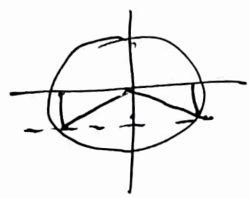
$$\boxed{y = \frac{5}{4} + \frac{3\sqrt{3}}{4} - x} \checkmark$$

horizontal & vertical tangents:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \quad \begin{array}{l} \leftarrow = 0: \ \frac{dy}{dx} = 0 \ \left(\text{but required } \frac{dx}{d\theta} \neq 0 \text{ or } \frac{0}{0}!\right) \\ \leftarrow = 0: \ \frac{dy}{dx} \to \pm\infty \ \left(\text{if } \frac{dy}{d\theta} \neq 0 \text{ or } \frac{0}{0}!\right) \end{array}$$

$$0 = y'(\theta) = \underbrace{\cos\theta}_{=0}\,\underbrace{(1+2\sin\theta)}_{=0}$$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\qquad \sin\theta = -\frac{1}{2}$

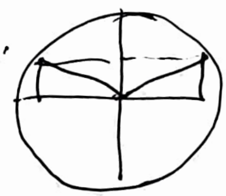


$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

(hor tan lines)

$$0 = x'(\theta) = \underbrace{(1+\sin\theta)}_{=0}\,\underbrace{(1-2\sin\theta)}_{=0}$$

$\theta = \frac{3\pi}{2}$ $\qquad \sin\theta = \frac{1}{2}$



$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

(vert tan lines)

trouble:

$$\theta = \frac{3\pi}{2}: \quad \frac{dy}{dx} = \frac{\overset{\to 0}{\cos\theta}}{\underset{\to 0}{1+\sin\theta}} \, \frac{\overset{\to -1}{(1+2\sin\theta)}}{\underset{\to +3}{(1-2\sin\theta)}} \sim \frac{\cos\theta}{1+\sin\theta}\left(-\frac{1}{3}\right)$$

$$\lim_{\theta\to\frac{3\pi}{2}^-} \frac{\cos\theta}{1+\sin\theta} = \lim_{\theta\to\frac{3\pi}{2}^-} \frac{-\sin\theta}{0+\cos\theta} = \lim_{\theta\to\frac{3\pi}{2}^-} -\tan\theta = -\infty$$

l'Hopital

$$\lim_{\theta\to\frac{3\pi}{2}} \frac{dy}{dx} = +\infty \checkmark$$

obvious

calculate $(x,y)$ for hor/vert tan lines

(Maple)