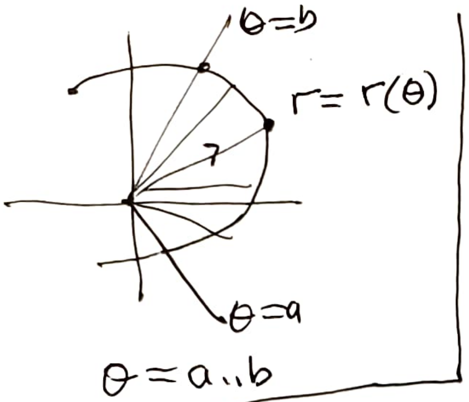
$\boxed{10.4c}$ Arclength for polar curves



← A polar curve is just a parametrized curve and we already developed a formula for these!

$$L=\int_a^b \underbrace{\sqrt{x'(t)^2+y'(t)^2}}\,dt \quad = \text{distance traveled}$$

$$\frac{ds(t)}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = v(t) \text{ speed if } t \text{ is time}$$

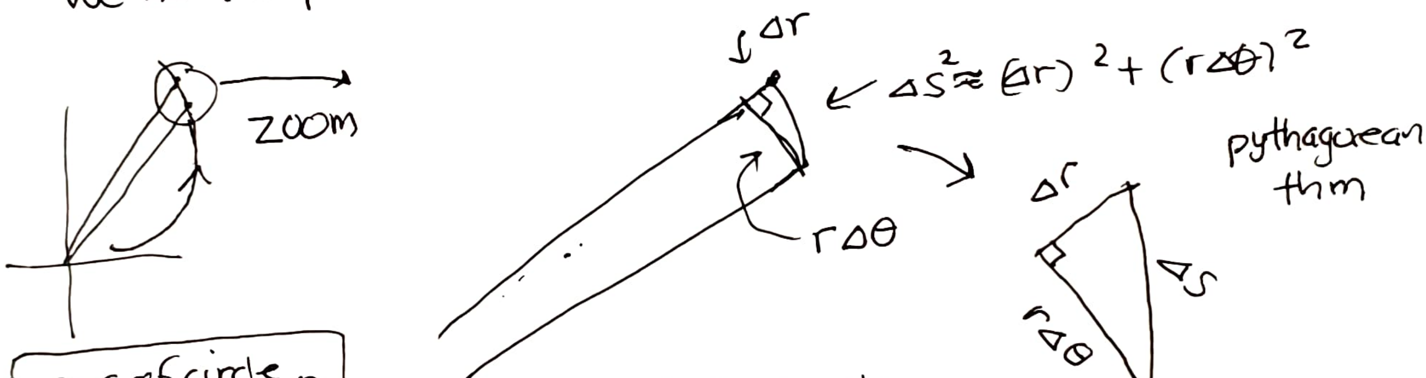
$$x(\theta) = r(\theta)\cos\theta \rightarrow x' = r'\cos\theta - r\sin\theta \qquad (x')^2 = (r')^2c^2 - 2rr'cs + r^2s^2$$

$$y(\theta) = r(\theta)\sin\theta \rightarrow y' = r'\sin\theta + r\cos\theta \qquad (y')^2 = (r')^2s^2 + 2rr'cs + r^2c^2$$

$$(x')^2+(y')^2 = (r')^2(c^2+s^2) + r^2(s^2+c^2) = (r')^2+r^2$$

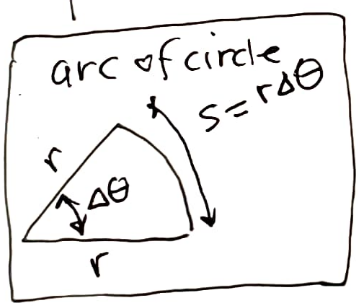
so $\boxed{L=\int_a^b\sqrt{r^2+r'^2}\,d\theta}$

BUT this is just manipulation of formulas without intuition.
BUT this is just intuitively obvious.
We need a picture to make this intuitive.



$\leftarrow \Delta s^2 \approx (\Delta r)^2 + (r\Delta\theta)^2$

pythagorean thm



In the limit $\Delta \to d$:

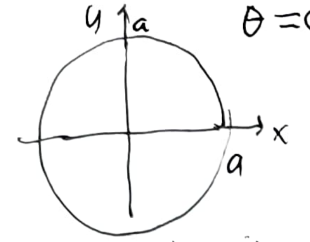
$$ds = \sqrt{dr^2+r^2d\theta^2} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 d\theta^2 + r^2 d\theta^2} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta$$

$$\boxed{s = \int_a^b ds(\theta) = \int_a^b \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta}$$

or $\boxed{L = \int_a^b \sqrt{r^2+r'^2}\,d\theta}$

10.4c Arclength of polar curves

example $r = a > 0$, $\theta = 0..2\pi$

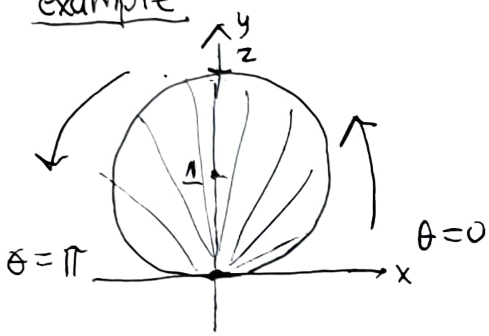


circle at origin

$$\frac{dr}{d\theta} = 0; \quad \sqrt{r^2 + r'^2} = \sqrt{a^2} = a$$

$$L = \int_0^{2\pi} a \, d\theta = a\theta \Big|_0^{2\pi} = 2\pi a \quad \checkmark$$

example



unit circle centered on axis

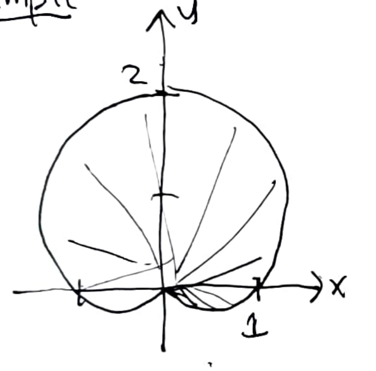
$r = 2\sin\theta$, $\quad \theta = 0..\pi$ (1 loop!)

$r' = 2\cos\theta$

$$r^2 + r'^2 = 4\sin^2\theta + 4\cos^2\theta = 4(\sin^2\theta + \cos^2\theta) = 4 \quad \text{constant!}$$

$$L = \int_0^{\pi} \sqrt{r^2 + r'^2} \, d\theta = \int_0^{\pi} 2 \, d\theta = 2\theta \Big|_0^{\pi} = 2\pi \quad \checkmark$$

example



cardioid
$\theta = 0..2\pi$

$r = 1 + \sin\theta$

$r' = \cos\theta$

$$r^2 + r'^2 = (1+\sin\theta)^2 + \cos^2\theta = 1 + 2\sin\theta + (\sin^2\theta + \cos^2\theta) = 2(1+\sin\theta)$$

$$L = \int_0^{2\pi} \sqrt{2(1+\sin\theta)} \, d\theta \underset{\text{Maple}}{=} 8$$

why so simple? perfect square thru trig identities!

$$\int_0^{2\pi} \sqrt{\frac{2 \cdot 2(1+\sin\theta)}{2}} \, d\theta = 2\int_0^{2\pi} \sqrt{\frac{1+\sin\theta}{2}} \, d\theta = 2\int_{-\pi/2}^{3\pi/2} \underbrace{\sqrt{\frac{1+\cos u}{2}}}_{|\cos u/2|} \, du$$

$$\sin\theta = \cos\left(\tfrac{\pi}{2} - \theta\right) = \cos(\underbrace{\theta - \pi/2}_{\equiv u})$$

$$= 2\int_{\underbrace{-\pi}}^{\pi} \cos\frac{u}{2} \, du \overset{\text{symmetry}}{=} 4\int_0^{\pi} \cos\frac{u}{2} \, du = 4\,\frac{\sin\frac{u}{2}}{\frac{1}{2}} \Big|_0^{\pi} = 8\sin\frac{\pi}{2} = 8 \quad \checkmark$$

part of circle where $\cos\frac{u}{2} \geq 0$

10.4c Arclength of polar curves

exercise: $r = \sin\frac{\theta}{4}$ $\quad \theta = 0 .. ?$

what is the period after which this curve retraces its path ?

$$L = \int_a^b \sqrt{r^2 + r'^2}\, d\theta$$

The sqrt makes it difficult to exactly integrate polar curves in terms of elementary functions or techniques of integration we know, so perfect squares or factoring out of the sqrt to get a simple u-sub is all we can do without the aid of Maple.