

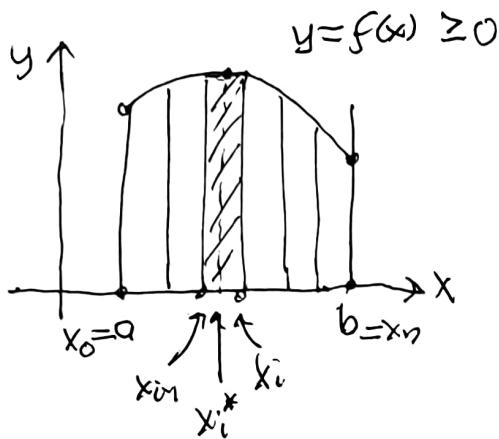
10.4b

Areas delimited by polar curves

①

Cartesian coordinates:

Riemann approach to area under curve



$$\Delta A_i \approx f(x_i^*) \Delta x \quad (\text{rectangle})$$

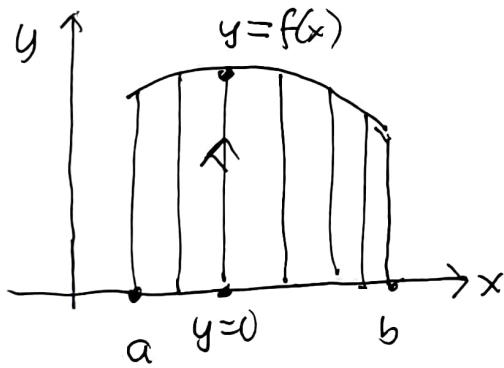
$$A \approx \sum_{i=1}^n f(x_i^*) \Delta x$$

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

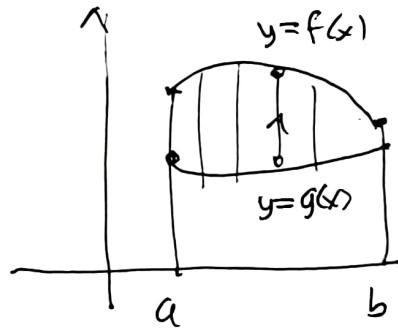
$$= \int_a^b f(x) dx \geq 0$$

↑ ordered $a < b$

symbolized by
diagram



generalize
to
area
between
curves



↑ typical linear
cross-section of
region
sweeps across from left to right
picking up all the area under
the curve

$$\begin{aligned} A &= \int_a^b f(x) dx - \int_a^b g(x) dx \\ &= \int_a^b \underbrace{f(x) - g(x)}_{\geq 0 \text{ to get positive area}} dx \geq 0 \end{aligned}$$

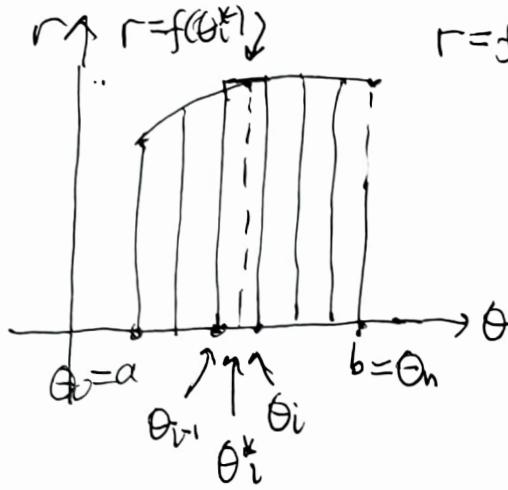
so f above g over interval

104b

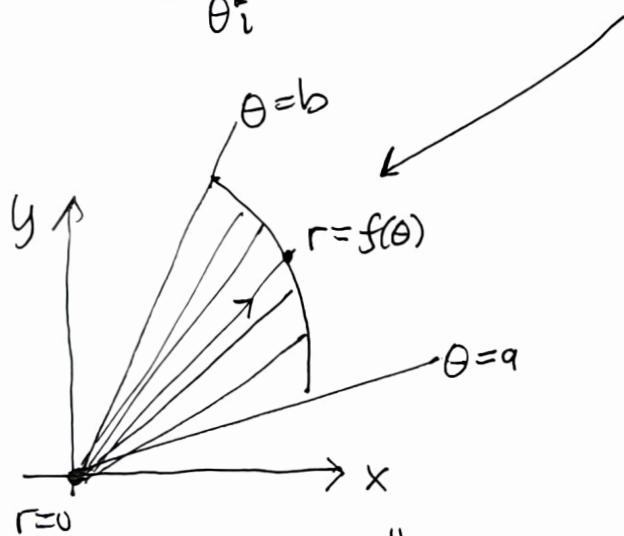
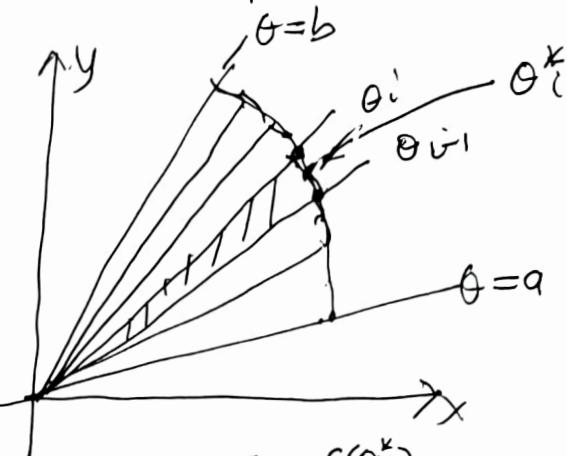
Areas delimited by polar curves

2

polar coords: Riemann approach to area "inside" polar curves

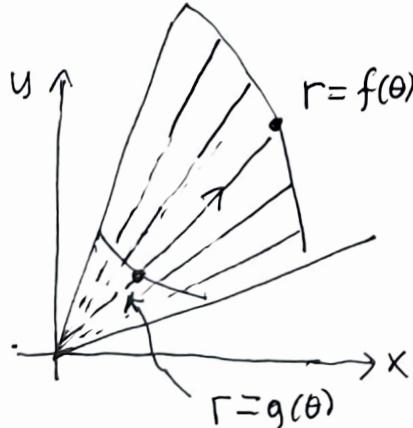


$$r = f(\theta) \geq 0$$



"radial cross-section"
from origin out to polar curve

Area is "inside" the polar curve
swept out as radial cross-section
moves through region
rotating around origin



sector area:

$$\frac{\Delta A}{A} = \frac{\Delta\theta}{2\pi}$$

fraction of pie
= fraction of circumference

$$\Delta A = A \frac{\Delta\theta}{2\pi} = \frac{(\pi r^2)}{2\pi} \Delta\theta$$

$$= \frac{1}{2} r^2 \Delta\theta$$

$$\text{so } \Delta A_i = \frac{1}{2} f(\theta_i^*)^2 \Delta\theta$$

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{2} f(\theta_i^*)^2 \Delta\theta$$

$$= \int_a^b \frac{1}{2} f(\theta)^2 d\theta \quad = \int_a^b \frac{1}{2} r^2 d\theta$$

$$A = \int_a^b \frac{1}{2} f(\theta)^2 d\theta - \int_a^b \frac{1}{2} g(\theta)^2 d\theta$$

$$= \int_a^b \underbrace{\frac{1}{2} (f(\theta)^2 - g(\theta)^2)}_{\geq 0} d\theta \geq 0$$

area "inside" $r=f(\theta)$ and "outside" $r=g(\theta)$
with respect to the radial direction

10.4b

Areas delimited by polar curves

(3)

video example : $r^2 = \sin 2\theta \geq 0$ limits domain of θ

\downarrow

$\sin 2\theta < 0$ complex r , no curve for these angles

\downarrow

$r = \pm \sqrt{\sin 2\theta}$ two polar curves moving in opposite quadrants

\uparrow

in opposite quadrants

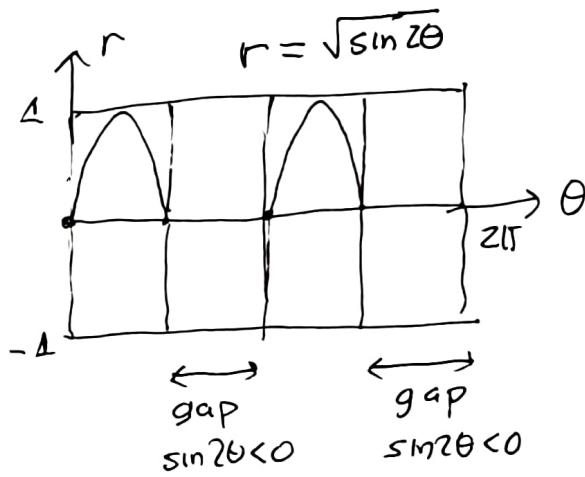
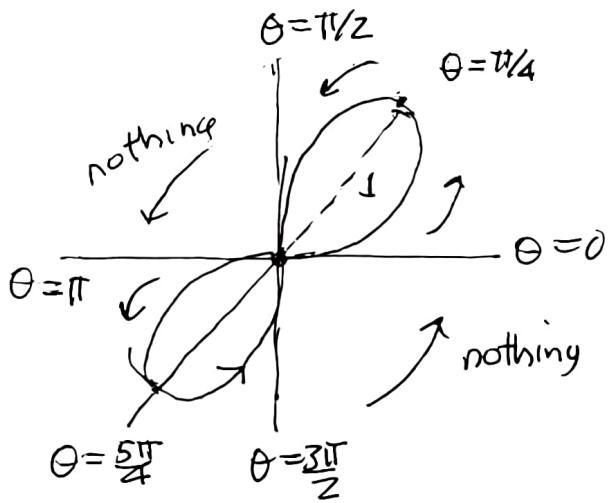
$\sin 2\theta \geq 0$ if $0 \leq 2\theta \leq \pi$ (upper half circle for $\sin \theta$)

\downarrow

so $0 \leq \theta \leq \frac{\pi}{2}$

\downarrow

$r = \sqrt{\sin 2\theta} \rightarrow r=0 \quad r>0 \quad r=0$ one loop



$$A_{\text{loop}} = \int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \int_0^{\pi/2} \frac{1}{2} (\sin 2\theta) d\theta = \frac{1}{2} \left(-\frac{1}{2} \cos 2\theta \right) \Big|_0^{\pi/2}$$

$$= -\frac{1}{4} (\cos \pi - \cos 0) = \frac{1}{4} = \frac{1}{2}$$

$$A_{2\text{loops}} = 2 \left(\frac{1}{2} \right) = 1$$

But if you don't think:

$$\int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} \sin 2\theta d\theta = -\frac{1}{4} \cos 2\theta \Big|_0^{2\pi}$$

$$= -\frac{1}{4} (\cos 4\pi - \cos 0) = 0$$

[cancellation from $r^2 < 0$]



traces out same loops but in opposite quadrants

10.4b

Areas delimited by polar curves

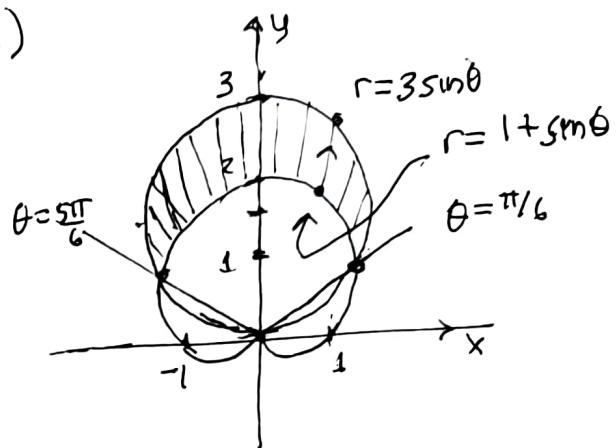
(4)

example

Find area inside $r = 3\sin\theta$ (circle)
 and outside $r = 1 + \sin\theta$ (cardioid)

intersection: $3\sin\theta = 1 + \sin\theta$

$$2\sin\theta = 1 \\ \sin\theta = \frac{1}{2} \rightarrow \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$



angular range for "shaded region"

$$\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6} \quad \text{combine}$$

$$A_{\text{diff}} = \int_{\pi/6}^{5\pi/6} \frac{1}{2} (3\sin\theta)^2 d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2} (1+\sin\theta)^2 d\theta$$

$$= \int_{\pi/6}^{5\pi/6} \frac{1}{2} (9\sin^2\theta - (1+2\sin\theta+\sin^2\theta)) d\theta$$

$$= \int_{\pi/6}^{5\pi/6} \frac{1}{2} (8\sin^2\theta - 2\sin\theta - 1) d\theta \underset{\text{Maple}}{=} \pi \approx 3.14$$

$$A_{\text{circle}} = \pi \left(\frac{3}{2}\right)^2 = \frac{9}{4}\pi \approx 7.07$$

$$\frac{A_{\text{diff}}}{A_{\text{circle}}} = \frac{\pi}{\frac{9}{4}\pi} = \frac{4}{9} \approx 0.44 \quad \text{a bit less than half, okay}$$

$$A_{\text{cardioid}} = \int_0^{\pi} \frac{1}{2} (1+\sin\theta)^2 d\theta \underset{\text{Maple}}{=} \frac{3\pi}{2} \approx 4.71$$

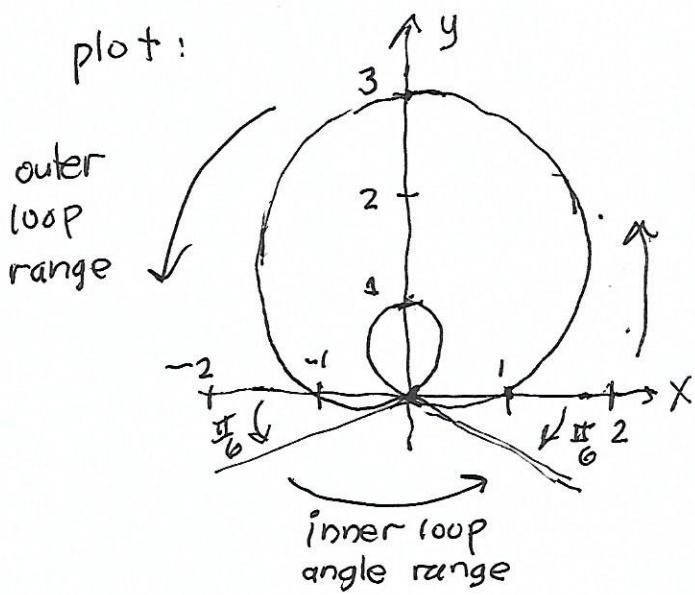
check Maple polar plot
to compare (with circle)

10.46 Areas delimited by polar curves

(5)

Ex Find the area in between the inner and outer loops of the limacon: $r = 1 + 2 \sin \theta$

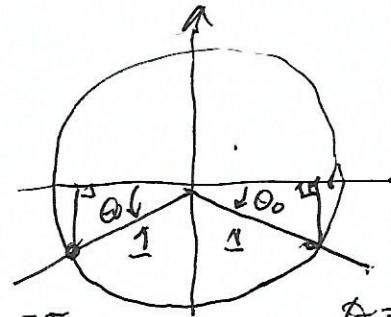
plot:



Find interval where $r < 0$:

$$0 = r = 1 + 2 \sin \theta$$

$$\sin \theta = -\frac{1}{2} \Leftrightarrow \text{reference angle } \theta = \pi/6: \\ \theta = -\pi/6$$



$$\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6} \\ \text{or } -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$$

$$\theta = -\pi/6 \\ \text{or} \\ 2\pi - \pi/6 \\ = 11\pi/6$$

We need a continuous interval $\theta_{\min} \leq \theta \leq \theta_{\max}$ for each part of the curve.

$$\text{inner loop } r < 0: \quad \theta = -\frac{5\pi}{6} \dots -\frac{\pi}{6} \quad \text{or} \quad \theta = -\frac{7\pi}{6} \dots \frac{11\pi}{6}$$

$$\text{outer loop } r > 0: \quad \theta = -\frac{\pi}{6} \dots \frac{7\pi}{6}$$

$$A_{in} = \int_{-\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2} (1 + 2 \sin \theta)^2 d\theta = -\frac{3\sqrt{3}}{2} + \pi \quad \text{Maple}$$

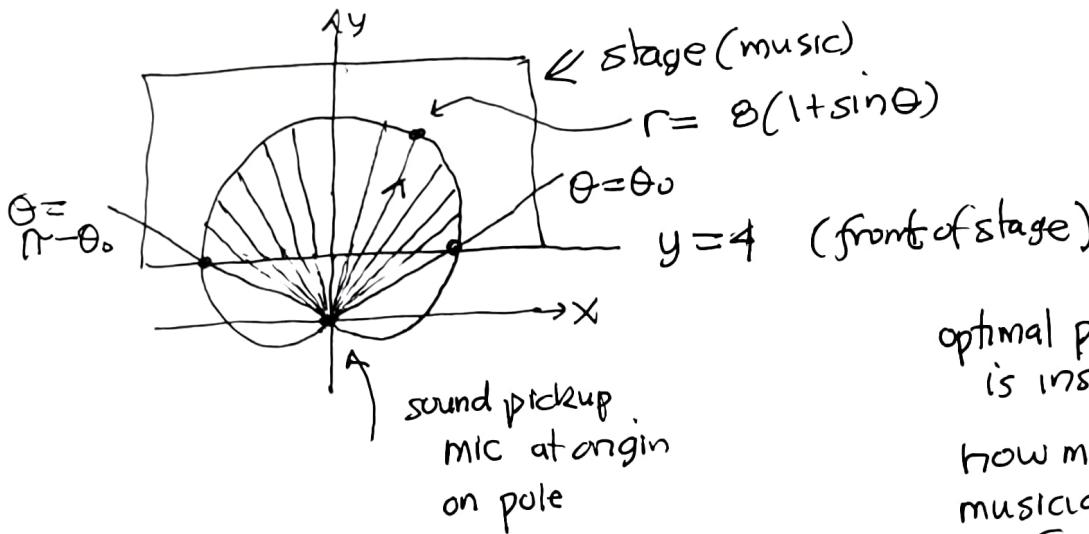
$$A_{out} = \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2} (1 + 2 \sin \theta)^2 d\theta = \frac{3\sqrt{3}}{2} + 2\pi \quad \text{Maple}$$

$$A = A_{out} - A_{in} = \left(\frac{3\sqrt{3}}{2} + 2\pi \right) - \left(-\frac{3\sqrt{3}}{2} + \pi \right) = 3\sqrt{3} + \pi$$

10.4b

Areas delimited by polar curves

(5)

exercise 10.4.44 "application"Find intersection points

$$4 = y = r \sin \theta \rightarrow r = \frac{4}{\sin \theta} = 4 \csc \theta \quad (\text{polar curve form})$$

$$\left. \begin{aligned} 4 \csc \theta &= 8(1 + \sin \theta) \\ \frac{4}{\sin \theta} &= 8 \sin \theta (1 + \sin \theta) = 8 \sin \theta + 8 \sin^2 \theta \\ 4 &= 8 \sin \theta + 8 \sin^2 \theta \Rightarrow 2 \sin^2 \theta + 2 \sin \theta - 1 = 0 \\ 2 \sin \theta &= \frac{-2 \pm \sqrt{4 - 4(2)(-1)}}{2(2)} = \frac{-1 \pm \sqrt{1+2}}{2} \\ &= \frac{-1 \pm \sqrt{3}}{2} \end{aligned} \right\} \begin{aligned} \sin \theta &= \frac{-1 + \sqrt{3}}{2} \quad (\text{neglect invalid}) \\ \theta_0 &= \arcsin \left(\frac{\sqrt{3}-1}{2} \right) \approx 0.3747344329 \end{aligned}$$

Integrate over $\theta = \theta_0 \dots \pi - \theta_0$:

$$A = \int_{\theta_0}^{\pi - \theta_0} \frac{1}{2} (8(1 + \sin \theta))^2 d\theta - \underbrace{\int_{\theta_0}^{\pi - \theta_0} \frac{1}{2} (4 \csc \theta)^2 d\theta}_{\text{triangle area}}$$

\approx Maple $40.68 \approx 40.7 \text{ m}^2$

In context numerical value is goal
(significant figures play a role)