10.4a Areas enclosed by polar curves : angular ranges (1)

Before we tackle the problem of how to evaluate areas enclosed by polar curves:

$$A = \int_{\theta_1}^{\theta_2} \frac{dA}{d\theta}\, d\theta \quad , \quad \frac{dA}{d\theta} \geq 0 \qquad \text{(we discuss } dA/d\theta \text{ next time)}$$

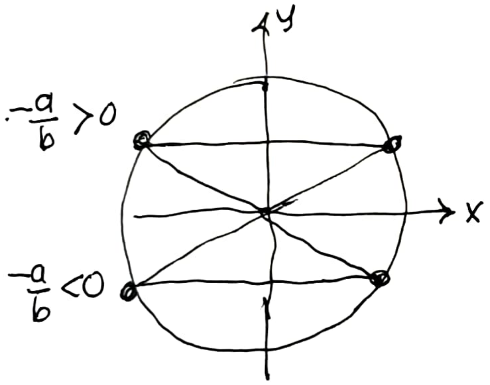
it is useful to first deal with the ranges of the radial and angular variables that describe regions of the plane bounded by polar curves which are multivalued functions of θ, with r values of both signs.

A $\boxed{\text{limacon}}$ with both an inner and outer loop is a good exercise to confront these complications that also occur in Cartesian coordinates when we deal with parametric curves, which is exactly the source of these complications.

consider these curves: $\boxed{r = a + b\sin\theta}$ with $|a| < |b|$

solve for zeros: $r = a + b\sin\theta = 0 \rightarrow \sin\theta = -\frac{a}{b}$

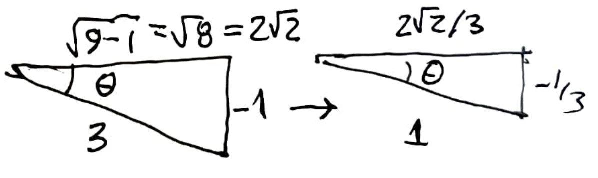
$|\sin\theta| < 1$ means 2 solutions on the unit circle for $0 \leq \theta \leq 2\pi$ or $-\pi \leq \theta \leq \pi$

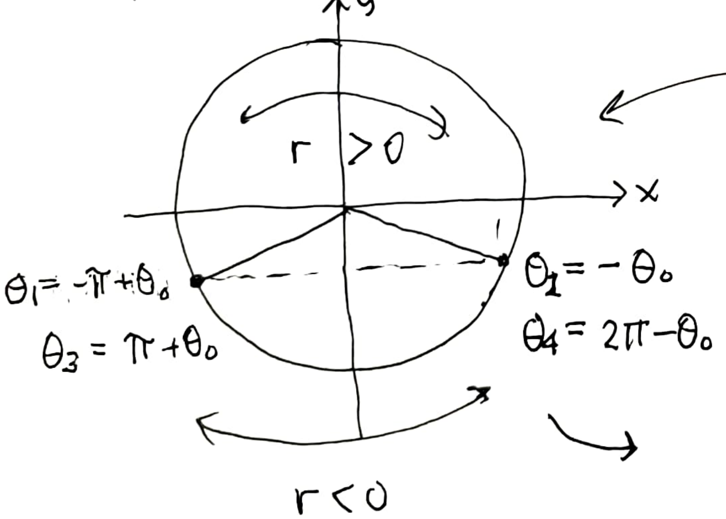


solutions reflect across the y-axis since $\sin(\pi - \theta) = \sin\theta$

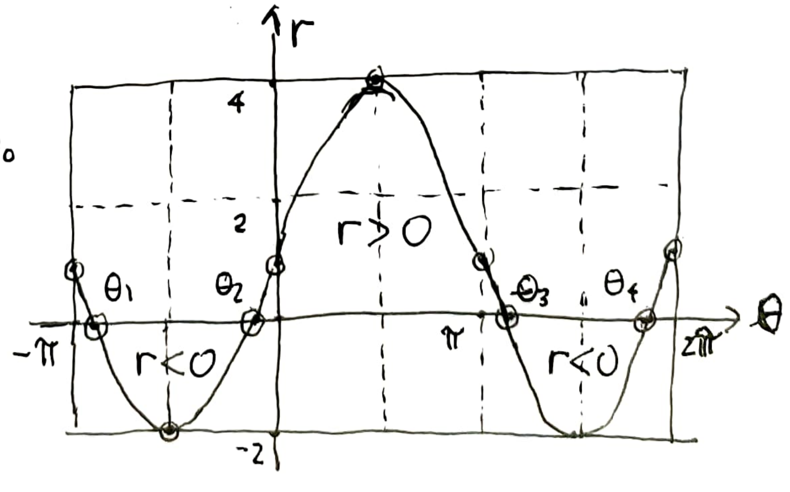
explicit example $\quad r = 1 + 3\sin\theta = 0 \rightarrow \sin\theta = -1/3 < 0$

$\theta_0 = \arcsin(1/3) \approx 19.4°$





r versus θ :



10.4a Areas enclosed by polar curves: angular ranges

$r = 1 + 3\sin\theta$ continued

If we trace this out by hand we have the following sequence of key points.

$\theta = -\pi$: $r = 1 + 0 = 1$

$\theta = \theta_1$: $r = 0$

$\theta = -\frac{\pi}{2}$: $r = 1 - 3 = -2$ — $r < 0$ — $\theta_1 \le \theta \le \theta_2$

$\theta = \theta_2$: $r = 0$

$\theta = 0$: $r = 1 + 0 = 1$

$\theta = \pi/2$: $r = 1 + 3 = 4$ — $r > 0$

$\theta = \pi$: $r = 1 + 0 = 1$

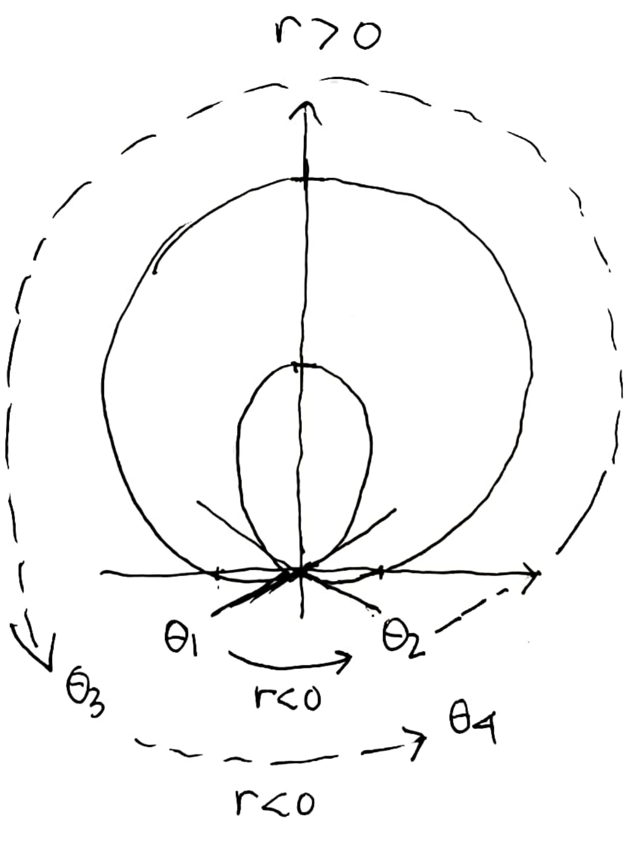
$\theta = \theta_3$: $r = 0$

$\theta = 3\pi/2$: $r = 1 - 3 = -2$ — $r < 0$ — $\theta_3 \le \theta \le \theta_4$

$\theta = \theta_4$: $r = 0$

$\theta = 2\pi$: $r = 1 + 0 = 1$

2 choices for inner loop



Easily drawn & animated in Maple

$r = 1 + 3\sin\theta$ continued

The integration interval for the outer loop $r \geq 0$ is unambiguous: $\quad -\theta_0 \leq \theta \leq \pi + \theta_0$

The inner loop has two choices for $r \leq 0$:

- before: $\quad -\pi + \theta_0 \leq \theta \leq -\theta_0$
- after: $\quad \pi + \theta_0 \leq \theta \leq 2\pi - \theta_0$

so

$$A_{outer} = \int_{-\theta_0}^{\pi+\theta_0} \frac{dA}{d\theta}\, d\theta$$

$$A_{inner} = \int_{-\pi+\theta_0}^{-\theta_0} \frac{dA}{d\theta}\, d\theta = \int_{\pi+\theta_0}^{2\pi-\theta_0} \frac{dA}{d\theta}\, d\theta$$

$$A_{diff} = A_{outer} - A_{inner}$$

exercise. Analyze $r = 1 - 3\cos\theta$ in the same way.