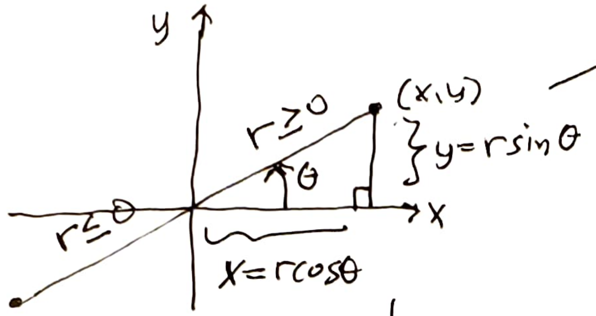
10.3 Polar coordinates

Polar coordinates (r, θ) are adapted to concentric circles around the origin instead of rectangles like the "rectangular" Cartesian coordinates (x, y). The trig of right triangles for acute angles in the first quadrant is automatically transferred to the other 3 quadrants.

Notation and definitions



from (r, θ) to (x, y):

$x = r\cos\theta$
$y = r\sin\theta$

$r \geq 0$ for unique radial coordinate

$0 \leq \theta < 2\pi$ for unique angular coordinate
or
$-\pi < \theta \leq \pi$

But $-\infty < r < \infty$, $-\infty < \theta < \infty$ for "polar" curves that wrap around the origin

from (x, y) to (r, θ):

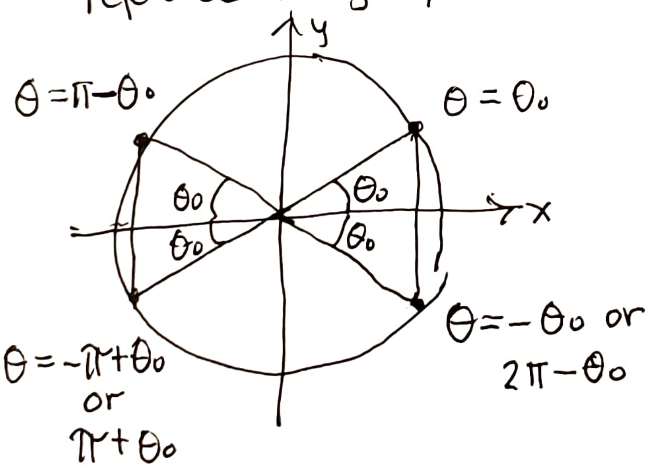
Pythag: $r^2 = x^2 + y^2$
$r = \sqrt{x^2+y^2}$ (normally)

$\tan\theta = \dfrac{y}{x}$ = slope of hypotenuse line

$\theta = \arctan\left(\dfrac{y}{x}\right) + {?}$ (constant)

$\arctan(u) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ works on right half of plane for $-\pi < \theta \leq \pi$, but need to add $\pm\pi, 2\pi$ on left half of plane, 4th quadrant for $0 \leq \theta < 2\pi$

reference triangles, acute reference angle θ_0



Maple: $\arctan(y, x) \in (-\pi, \pi]$

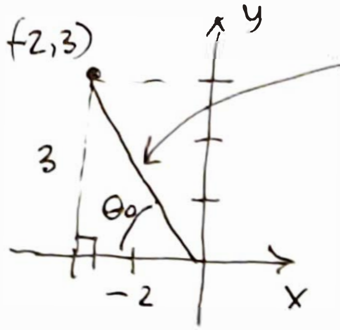
new function that produces a unique angle in this interval for all 4 quadrants.

$\boxed{10.3}$ Polar coordinates $\boxed{\text{point coordinates}}$

From Cartesian to polar

no formulas needed! draw diagram

example $(x,y) = (-2,3) \longrightarrow (r,\theta) = \ ?$



$r = \sqrt{9+4} = \sqrt{13}$ pythag thm!

$\theta_0 = \arctan 3/2 \longrightarrow \theta = \pi - \theta_0 = \underbrace{\pi - \arctan 3/2}$

STOP.

This is the "exact angle".
We can approximate to a decimal in radians, but for interpretation, convert to an angle in degrees. In math formulas ALWAYS radians.

$\theta \approx 2.1588$ (meaningless to us)

$\approx 123.7°$ ($\approx 56.3°$ from horizontal)

conclusion: $(r,\theta) = (\sqrt{13}, \pi - \arctan 3/2)$

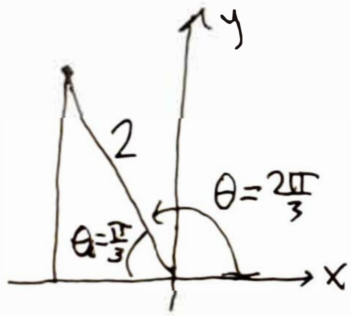
From polar to Cartesian

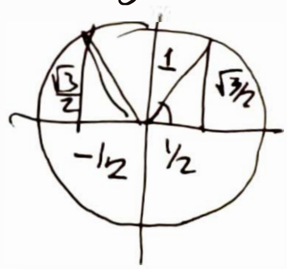
$(r,\theta) = (2, +2\pi/3) \longrightarrow (x,y) = \ ?$

example: Draw diagram

$\pi/3$ less than π

reference angle $\theta_0 = \pi/3$:





$\cos\frac{\pi}{3} = 1/2$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{2\pi}{3} = -1/2$

$\sin 2\pi/3 = \sqrt{3}/2$

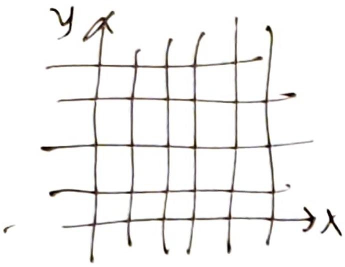
$$\begin{cases} x = r\cos\theta = 2(-1/2) = -1 \\ y = r\sin\theta = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3} \end{cases}$$

Conclusion $(x,y) = (-1, \sqrt{3})$

yes, we can just "plug in" but we also want to visualize where we are in the plane, and what the reference angle is.

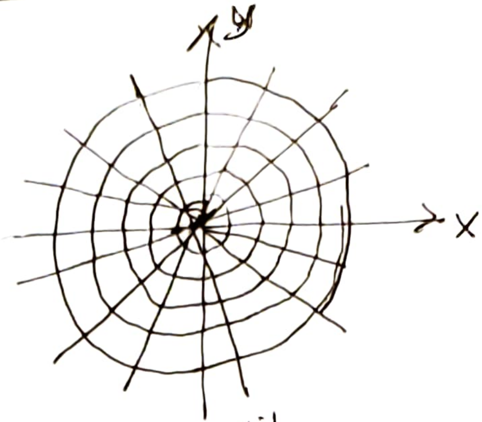
10.3 Polar coordinates

Functional relationships: polar curves



Cartesian "rectangular" coord grid.

constant x lines
constant y lines
(equally spaced)



polar coord grid.

constant r circles
constant θ half lines (rays)
(equally spaced)

We use function graphs in both "directions" here:

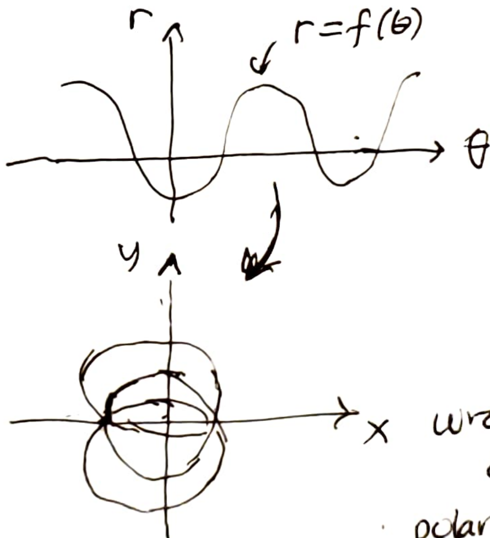
$y=f(x)$ or $x=g(y)$

"independent" variable

Here

$r=f(\theta)$ or $\cancel{\theta=g(r)}$

no, not useful



polar "coordinate" plane

physical x-y plane

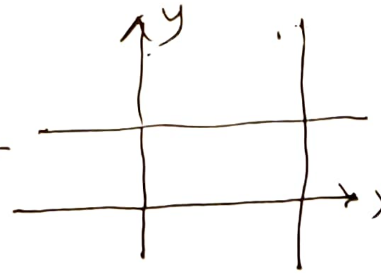
wraps around origin

polar curves defined for interval wider than 2π — often multiples of 2π or even infinite.

10.3 Polar coordinates

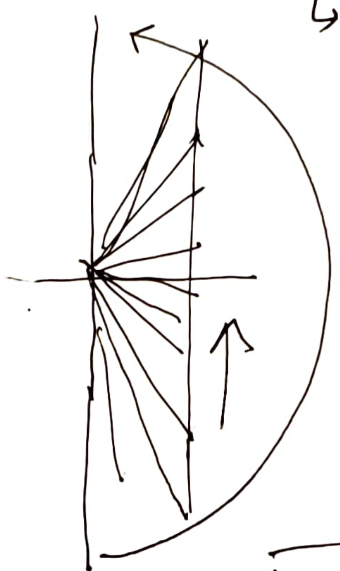
simplest curves : straight lines : horizontal and vertical

Express these as polar curves:
$r = f(\theta)$



$3 = x = r\cos\theta$ solve for r

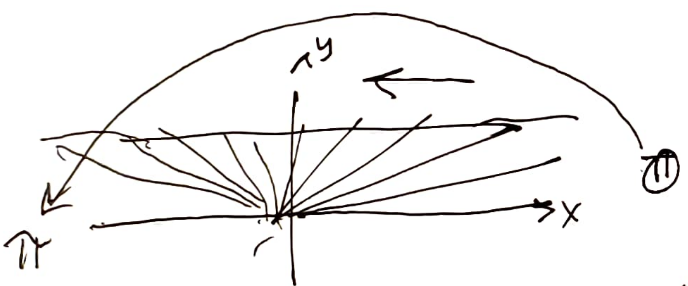
$\hookrightarrow r = \frac{3}{\cos\theta} = 3\sec\theta$



sweep θ rays
from $-\frac{\pi}{2}$ $(y \to -\infty)$
to $\pi/2$ $(y \to \infty)$
so
$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

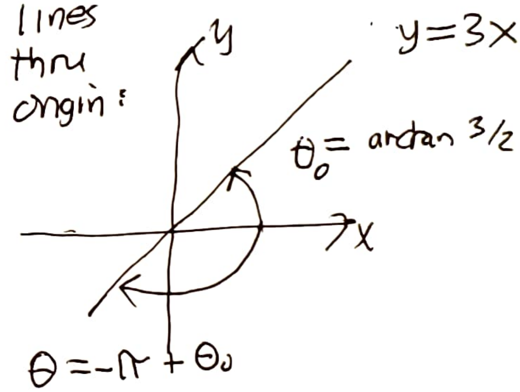
$2 = y = r\sin\theta$

$r = \frac{2}{\sin\theta} = 2\csc\theta$

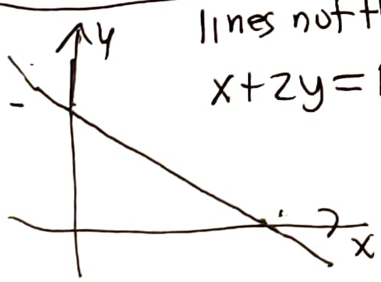


sweep θ rays from $\theta = 0$ $(x \to \infty)$
to $\theta = \pi$ $(x \to -\infty)$
so $0 < \theta < \pi$.

lines thru origin:



lines not thru origin or hor/vert:



$x + 2y = 1 \longrightarrow r\cos\theta + 2r\sin\theta = 1$

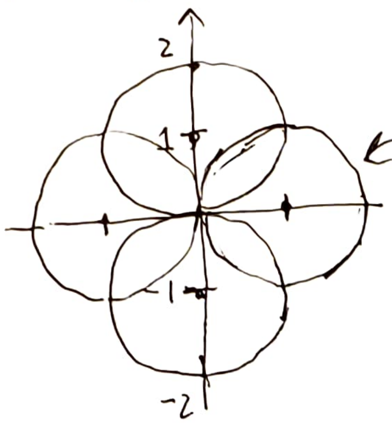
$r(\cos\theta + 2\sin\theta) = 1$

$r = \frac{1}{\cos\theta + 2\sin\theta}$

ugly, polar coords simply NOT USEFUL for such lines circles? 1) center at origin easy: $r = \text{constant}$

2) thru origin with center on horizontal or vertical axis

Four possibilities



unit radius

center 1 unit from origin

example: radius 1, center at $(1,0)$

standard eqn:

$$(x-1)^2+(y-0)^2=1$$

convert to polar coords

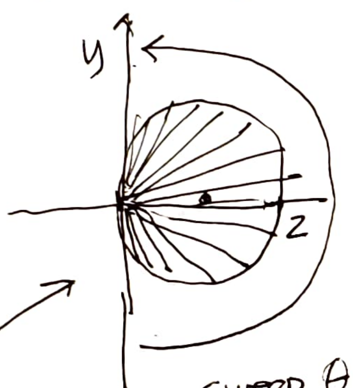
$$x^2-2x+\cancel{1}+y^2=\cancel{1}\,0$$

$$\underbrace{x^2+y^2}_{r^2}-\underbrace{2x}_{r\cos\theta}=0$$

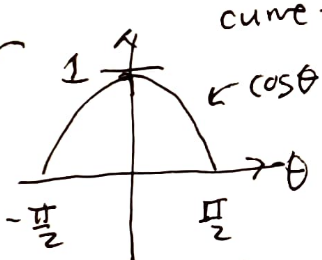
$$r^2-2r\cos\theta=0$$

$$r(r-2\cos\theta)=0$$

$$r=0 \text{ or } \boxed{r=2\cos\theta}$$



sweep θ ray from $\theta=-\pi/2$ to $\theta=\pi/2$, trace out curve exactly once, no repetition.



$\theta=0..\frac{\pi}{2}$, $\cos\theta=1..0$

$r=2\cos\theta=2\cos(-\theta)$ even function reflects across x-axis

start at $r=1+1=2$, decrease to $r=0+1=1$

so makes a loop (closed curve)

but starting from $r=2\cos\theta$ we dont know it represents a circle.

starting here:

$r=2\cos\theta$ transform to x-y?

$$\underbrace{r^2}_{x^2+y^2}=2\underbrace{r\cos\theta}_{x} \rightarrow x^2+y^2=2x$$

$$x^2-2x+y^2=0$$

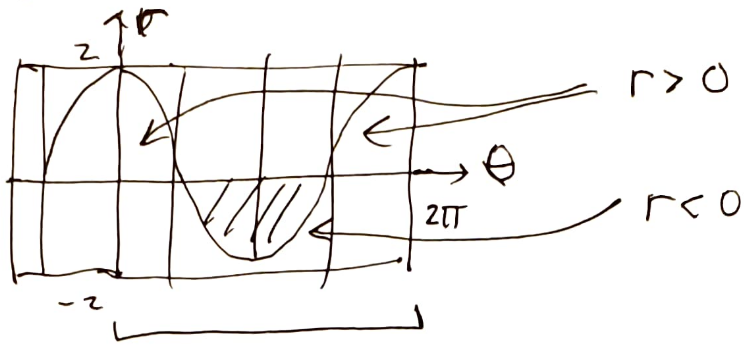
$$(x-1)^2+y^2=1$$

recognize as eqn of circle!

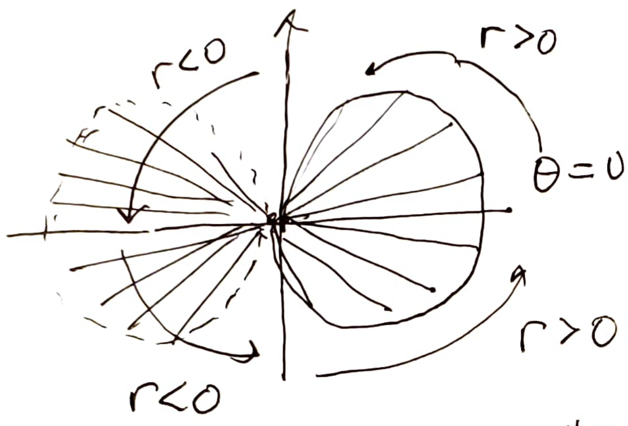
Similarly $r=2\sin\theta$ is the upper circle, symmetric across y-axis

Polar coordinates

$r = 2\cos\theta$ (continued)



$0 \le \theta < 2\pi$

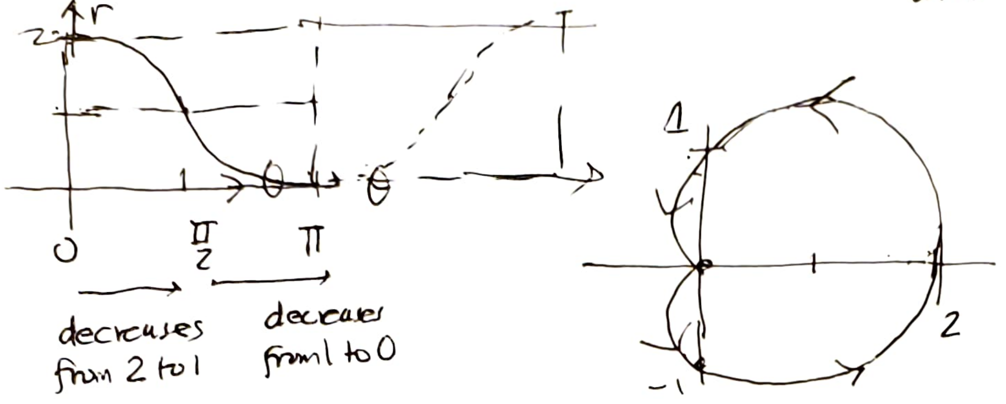


r>0: θ in Quad 1: upper half traced out
r<0: θ in Quad 2: lower half traced out
r<0: θ in Quad 3: upper half traced out
r>0: θ in Quad 4: lowe half traced out

$\theta = 0 .. 2\pi$ traces out curve twice!

When we are integrating over θ over a region of the plane, we need a unique angular range to describe it so we get no repetition in the integration process example Cardioid. $\leftarrow$ special case of limaçon: $r = a + b\cos\theta$ or $r = a + b\sin\theta$ with $|a| = |b|$

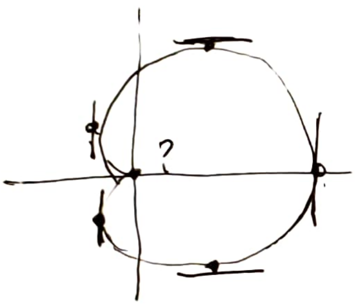
$r = 1 + \cos\theta$



decreases from 2 to 1 | decreases from 1 to 0

continues around counterclockwise reflected across x-axis

traces out this curve once: $0 \le \theta \le 2\pi$



horizontal and vertical tangent lines obvious
limiting tangent lines at origin "sharp point" is not obvious, requires limit

How to analyze slope? $\frac{dy}{dx}$? $\rightarrow$ tan line: $y - y_0 = m(x - x_0)$

Given $r = r(\theta)$ plug into:

$$\left.\begin{aligned} x &= r\cos\theta = r(\theta)\cos\theta \\ y &= r\sin\theta = r(\theta)\sin\theta \end{aligned}\right\}$$

this is just a parametrized curve with parameter θ! we know how to do this: $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$x = (1+\cos\theta)\cos\theta \rightarrow \frac{dx}{d\theta} = \frac{d}{d\theta}(\cos\theta + \cos^2\theta) = -\sin\theta + 2\cos\theta(-\sin\theta) = -\sin\theta(1 + 2\cos\theta)$

$y = (1+\cos\theta)\sin\theta \rightarrow \frac{dy}{d\theta} = \frac{d}{d\theta}(\sin\theta + \sin\theta\cos\theta) = \cos\theta + \cos^2\theta - \sin^2\theta$

$$\frac{dy}{dx} = \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta(1 + 2\cos\theta)}$$

For example $\theta = \pi/6$, $\cos\pi/6 = \sqrt{3}/2$, $\sin\pi/6 = 1/2$

$r = 1 + \sqrt{3}/2$ $\quad x = (1 + \sqrt{3}/2)\sqrt{3}/2 = x_0$
$\quad y = (1 + \sqrt{3}/2)\, 1/2 = y_0$

$$\left.\frac{dy}{dx}\right|_{\theta = \pi/6} = \frac{\sqrt{3}/2 + \underbrace{3/4 - 1/4}_{+1/2}}{-1/2(1 + 2\sqrt{3}/2)} = \frac{\sqrt{3}/2 + 1/2}{-1/2(1+\sqrt{3})} = -1 = m$$

so tan line:

$$y = \tfrac{1}{2}(1 + \tfrac{\sqrt{3}}{2}) - 1(x - \tfrac{\sqrt{3}}{2}(1 + \sqrt{3}/2))$$