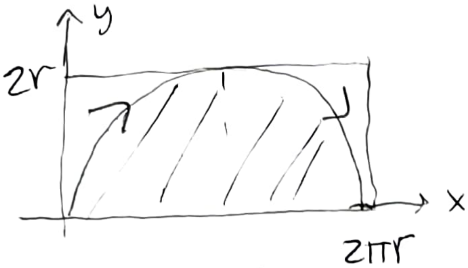
Calculus with parametrized curves



Area under curve?

$$A = \int_{x=0}^{x=2\pi r} y\, dx = \int_{t=0}^{t=2\pi} \underbrace{y(t)}_{\geq 0}\, \underbrace{dx(t)}_{\geq 0}$$ signs right.

$dx(t) \to \frac{dx}{dt}(t)\,dt$

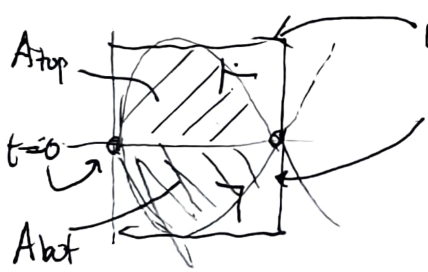
$$= \int_0^{2\pi} r(1-\cos t)\cdot\left(r(1-\cos t)\,dt\right)$$

$$= r^2 \underbrace{\int_0^{2\pi} (1-\cos t)^2\,dt}_{\text{Maple: } 3\pi} = 3\pi r^2$$

$$A_{\text{rectangle}} = (2r)(2\pi r) = 4\pi r^2$$

$$\frac{A}{A_{\text{rectangle}}} = \frac{3\pi r^2}{4\pi r^2} = \frac{3}{4}$$ looks about right. ✓

Back to first example area enclosed by loop: $x = t^2,\ y = t^3 - 3t$



upper: $t = -\sqrt{3}..0$, $y > 0$ but $\frac{dx}{dt} < 0$

lower: $t = 0..\sqrt{3}$, $y < 0$ but $\frac{dx}{dt} > 0$

$A = 2A_{\text{top}} = 2A_{\text{bot}}$
by symmetry

$$A_{\text{top}} = \int_{t=-\sqrt{3}}^{t=0} \underbrace{y}_{\geq 0}\, \underbrace{\left(-\frac{dx}{dt}\,dt\right)}_{\geq 0}$$

$$= \int_{-\sqrt{3}}^{0} (t^3 - 3t)(-2t\,dt)$$

$$= -2\int_{-\sqrt{3}}^{0} t^4 - 3t^2\,dt = -2\left(\frac{t^5}{5} - \frac{3t^3}{3}\right)\Big|_{-\sqrt{3}}^{0}$$

$$= -2\left(\frac{t^2}{5} - 1\right)t^3\Big|_{-\sqrt{3}}^{0} = +2\underbrace{\left(\frac{3}{5}-1\right)}_{=-\frac{2}{5}}(-3\sqrt{3})$$

$$= \frac{12\sqrt{3}}{5} \approx 4.16$$

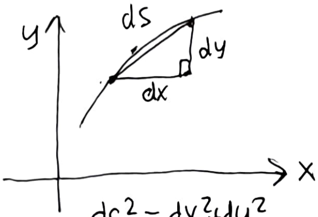
$A_{\text{rectangle}} = 4\cdot 3 = 12$

↕

$A = \frac{24\sqrt{3}}{5} \approx 8.32$
about right

$\boxed{10.2}$ Calculus with parametrized curves

Arclength?



Recall the arclength formula

$$L=\int_{x_1}^{x_2}\underbrace{\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

$$\sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)dx^2}=\sqrt{dx^2+dy^2}$$

$$\downarrow$$

$$dx(t)^2+dy(t)^2$$

$ds^2=dx^2+dy^2$
limiting Pythagorean thm

$$\int_{t=a}^{t=b}\sqrt{dx(t)^2+dy(t)^2}=\int_{t=a}^{t=b}\sqrt{\left(\frac{dx}{dt}dt\right)^2+\left(\frac{dy}{dt}dt\right)^2}$$

$$\boxed{=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=L}$$

$$L=\int_a^b\frac{ds}{dt}\,dt$$

one cycle of cycloid

$x=r(t-\sin t)\qquad \frac{dx}{dt}=r(1-\cos t)$

$y=r(1-\cos t)\qquad \frac{dy}{dt}=r\sin t$

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=r^2(1-\cos t)^2+r^2\sin^2 t$$
$$=r^2\left[1-2\cos t+\underbrace{(\cos^2 t+\sin^2 t)}_{1}\right]$$
$$=r^2\left[2-2\cos t\right]$$
$$=2r^2(1-\cos t)\quad\geq 0$$

$$L=\int_0^{2\pi}\sqrt{2r^2(1-\cos t)}\,dt=(\sqrt{2}\,r)\underbrace{\int_0^{2\pi}\sqrt{1-\cos t}\,dt}=8r$$

Maple: $4\sqrt{2}$

simple result because
perfect square using
half angle identity

$$\sqrt{1-\cos t}=\sqrt{2}\sqrt{\frac{1-\cos t}{2}}=\sqrt{2}\sqrt{\sin^2 t/2}=\sqrt{2}\sin t/2\quad \text{etc.}$$