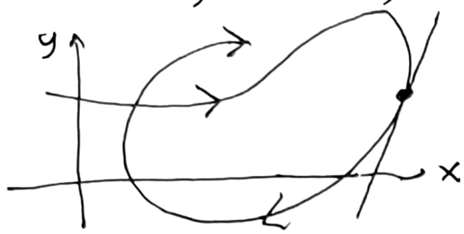
$\boxed{10.2}$ Calculus with parametrized curves: differentiation, integration, arclength

$x = x(t),\ y = y(t),\quad a \le t \le b$



To analyze tangent lines and concavity up or down for a parametrized curve we need to calculate $\frac{dy}{dx}, \frac{d^2y}{dx^2}$

where $x = x(t)$ and $y = y(t)$.
The chain rule enables us to convert x derivatives to t derivatives.

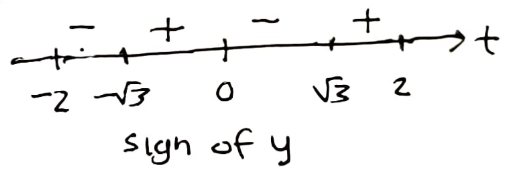
slope $$\frac{dy}{dx} = \frac{dy(t)}{dx(t)} = \frac{\frac{dy(t)}{dt}dt}{\frac{dx(t)}{dt}dt} = \frac{\frac{dy(t)}{dt}}{\frac{dx(t)}{dt}} = \frac{dy}{dx}(t)$$ quotient of differentials makes sense!

replacing y by dy/dx leads to the second derivative

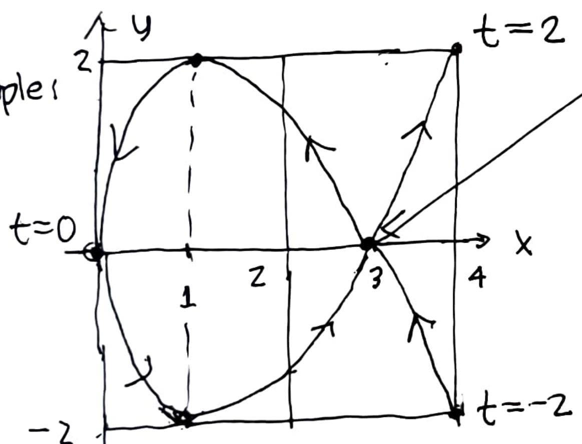
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}(t)\right)}{\frac{dx(t)}{dt}}$$ but we do this in 2 successive steps, not in one ugly formula!

Example $x = t^2$ (≥ 0), $y = t^3 - 3t$, $t = -2..2$

$= t(t^2-3) = 0 \to t = 0, \pm\sqrt{3}$ } x-intercepts
$x = 0, 3$



Maple:



endpoints: $t = \pm 2$
$x = (\pm 2)^2 = 4$
$y = \pm 2(4-3) = \pm 2$
(upper / lower)

slope: $x = t^2$ $\quad \frac{dx}{dt} = 2t$
$y = t^3 - 3t$ $\quad \frac{dy}{dt} = 3t^2 - 3 = 3(t^2-1)$ } $\frac{dy}{dx} = \frac{3(t^2-1)}{2t}$

horizontal tan lines:
$= 0 \to t = \pm 1$ $\quad t(t^2-3)$
$x = 1, y = \pm 1(1-3)$
$= \mp 2$

or vertical tan lines: $\frac{dx}{dy} = \frac{2t}{3(t^2-1)} = 0 \to t = 0$
$x = 0 = y$

concavity: $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{3(t^2-1)}{2t}\right)}{2t} = \frac{\frac{d}{dt}\left(\frac{3}{2}(t-t^{-1})\right)}{2t} = \frac{\frac{3}{2}(1+t^{-2})}{2t} = \frac{3}{4t^3}\underbrace{(t^2+1)}_{>0}$$
or quotient rule

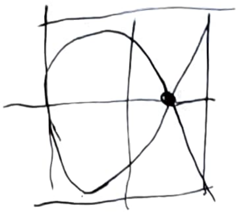
changes sign at $t = 0$ → $x = 0 = y$ inflection pt at $(0,0)$
$d^2y/dx^2 < 0$ when $t < 0$
> 0 when $t > 0$

 Calculus with parametrized curves

continued

tan lines at cross-over intersection point:

$$t = \pm\sqrt{3},\ (x, y) = (3, 0)$$



$$\frac{dy}{dx} = \frac{3(t^2-1)}{2t}$$

$$\left.\frac{dy}{dx}\right|_{t=\pm\sqrt{3}} = \frac{3(\cancel{3}^{\,1}-1)}{\cancel{2}(\pm\sqrt{3})} = \pm\sqrt{3} = m \quad \text{slope}$$

$$y - y_0 = m(x - x_0) \rightarrow \quad y - 0 = \pm\sqrt{3}(x-3)$$

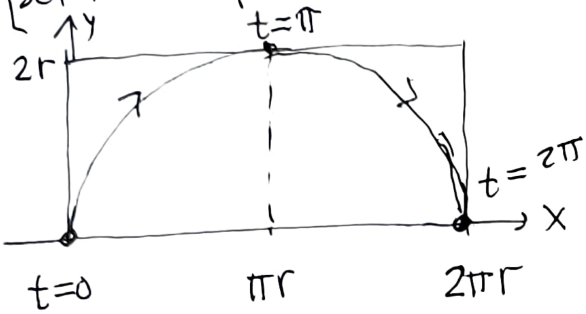
$$y = \pm\sqrt{3}(x-3)$$

plot with curve
(paste two expressions onto curve)

10.2 Calculus with parametrized curves

Example cycloid $x = r(t - \sin t)$, $y = r(1 - \cos t)$, $r > 0$ constant

[set $r = 1$ to plot, like measuring x and y in multiples of r]



$$x = r(t - \sin t),\quad \frac{dx}{dt} = r(1-\cos t)$$
$$y = r(1-\cos t),\quad \frac{dy}{dt} = r\sin t$$
$$\frac{dy}{dx} = \frac{\cancel{r}\sin t}{\cancel{r}(1-\cos t)} = \frac{\sin t}{1-\cos t}$$

hor tangents $\frac{dy}{dx} = 0 \to 0 = \sin t \to t = 0, \pi, 2\pi$

But $1 - \cos t = 0, \underbrace{1-(-1)}_{2}, 0$ — 0/0 limit!

ok: $t = \pi \to \boxed{x = \pi r,\ y = 2r}$ yes!

L'Hopital

$$\lim_{t\to 0^+} \frac{dy}{dx} = \lim_{t\to 0^+} \frac{\sin t}{1-\cos t} = \lim_{t\to 0^+} \frac{\cos t \nearrow 1}{\sin t \searrow 0^+} = \infty$$

$$\lim_{t\to 2\pi^-} \frac{dy}{dx} = \lim_{t\to 2\pi^-} \frac{\sin t}{1-\cos t} = \lim_{t\to 2\pi^-} \frac{\cos t \nearrow 1}{\sin t \searrow 0^-} = -\infty$$

limiting tangent lines but no tangent lines since 1-sided limits only exist. (infinite jump)

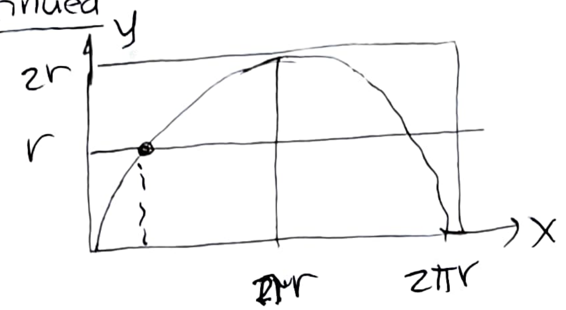
concavity

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{\sin t}{1-\cos t}\right)}{r(1-\cos t)} = \frac{\frac{(1-\cos t)(\cos t) - \sin t(\sin t)}{(1-\cos t)^2}}{r(1-\cos t)} = \frac{\cos t \underbrace{- \cos^2 t - \sin^2 t}_{-1}}{r(1-\cos t)^3}$$

$$= -\frac{(1-\cos t)}{r(1-\cos t)^3} = -\frac{1}{r(1-\cos t)^2} < 0 \text{ always concave down}$$

10.2 parametrized curves calculus

continued



tan lne where first crosses midway line

$y = r(1-\underbrace{\cos t}_{=0}) = r$

$=0 \to t = \pi/2$

(lowest soln!)

$x = r(t - \sin t)$

$= r\left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right) = r\left(\frac{\pi}{2} - 1\right)$

$$\frac{dy}{dx} = \frac{\sin t}{1-\cos t}$$

$$\left.\frac{dy}{dx}\right|_{t=\pi/2} = \frac{\sin \pi/2}{1-\cos\pi/2} = \frac{1}{1-0} = 1 = m$$

$(y - y_0) = m(x - x_0)$

$y - r = 1\left(x - (r)\left(\frac{\pi}{2} - 1\right)\right)$

$y = x + r + r\left(1 - \frac{\pi}{2}\right) = x + 2r - r\frac{\pi}{2}$

set $r=1$ to plot and check.