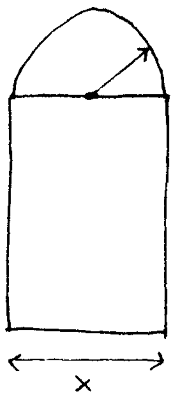
domain of a function is important



A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 ft, express the area A of the window as a function of the width x of the window

This is a nice problem because it shows how domain and range are very important in a simple geometry problem. All physical dimensions must be nonnegative and this leads to a finite interval for the domain and range here.

Let y be the height of the rectangular part of the window and let $r = x/2$ be the radius of the semicircle.

$$\left.\begin{aligned} C_\frown &= \tfrac{1}{2}(2\pi r) = \pi r = \frac{\pi x}{2} \\ C_\sqcup &= x + 2y \end{aligned}\right\} \quad \underset{\substack{\uparrow \\ \text{total} \\ \text{perimeter}}}{P} = C_\frown + C_\sqcup = \boxed{x + 2y + \frac{\pi x}{2} = 30}$$

Circumpherence formulas

constraint on x & y, only one is independent, solve for one of two and eliminate.

$$2y = 30 - x - \frac{\pi}{2}x$$

$$\boxed{y =} \boxed{\begin{aligned} 15 - \tfrac{1}{2}\left(1+\tfrac{\pi}{2}\right)x &\geq 0 \\ x &\geq 0 \end{aligned}}$$

constraints on x

domain:

$$0 \leq x \leq \frac{30}{1+\frac{\pi}{2}} \approx 11.67$$

$$A_\frown = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = \frac{\pi x^2}{8}$$

$$A_\square = xy = x\left(15 - \tfrac{1}{2}\left(1+\tfrac{\pi}{2}\right)x\right) = 15x - \tfrac{1}{2}\left(1+\tfrac{\pi}{2}\right)x^2$$

$$A = A_\frown + A_\square = 15x \underbrace{- \tfrac{1}{2}\left(1+\tfrac{\pi}{2}\right)x^2 + \tfrac{\pi}{8}x^2}_{\left(-\frac{1}{2} - \frac{\pi}{4} + \frac{\pi}{8}\right)x^2 = \left(-\frac{1}{2} - \frac{\pi}{8}\right)x^2 = -\frac{1}{2}\left(1+\frac{\pi}{4}\right)x^2} = 15x - \tfrac{1}{2}\left(1+\tfrac{\pi}{4}\right)x^2$$

so $A(x) = 15x - \frac{1}{2}\left(1+\frac{\pi}{4}\right)x^2$ ← just a quadratic function, graph is downturned parabola

$\quad\;\; = x\left(15 - \frac{1}{2}\left(1+\frac{\pi}{4}\right)x\right)$ ← crosses x-axis at $x = 0,\ \dfrac{30}{1+\frac{\pi}{4}} \approx 16.80$

peak of parabola at halfway point: $\dfrac{15}{1+\frac{\pi}{4}} \approx 8.40$

maximum area: $A = \dfrac{225}{2\left(1+\frac{\pi}{4}\right)} \approx 63.01$