Linear approximation versus differential approximation

Approximate \( \sin 29^\circ = \sin \left( \frac{29 \pi}{180} \right) \).

29° is close to 30° = \( \frac{30 \pi}{180} = \frac{\pi}{6} \), where we know the sine and cosine exactly, so either approximation should use 30° as the reference point.

**Linear approximation**

\[
y = f(x) = \sin x
\]
\[
f'(x) = \cos x
\]

\[
y = f\left(\frac{\pi}{6}\right) = \sin \frac{\pi}{6} = \frac{1}{2}
\]
\[
f'\left(\frac{\pi}{6}\right) = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}
\]

Write eq. of tan line:

\[
y - \frac{1}{2} = \frac{\sqrt{3}}{2} \left( x - \frac{\pi}{6} \right)
\]

Solve:

\[
y = \frac{1}{2} + \frac{\sqrt{3}}{2} \left( x - \frac{\pi}{6} \right)
\]

Linear approximation

\[
L(x) = \frac{1}{2} + \frac{\sqrt{3}}{2} \left( x - \frac{\pi}{6} \right)
\]

Now at \( x = 29^\circ = \frac{29 \pi}{180} \):

\[
f\left(29^\circ\right) \approx L\left(29^\circ\right) = \frac{1}{2} + \frac{\sqrt{3}}{2} \left( \frac{29 \pi}{180} - \frac{30 \pi}{180} \right)
\]

\[
= 0.50000 - 0.01511
\]

\[
= 0.48489 \quad \text{(done)}
\]

Compare with:

\[
f\left(29^\circ\right) = \sin 29^\circ = 0.48481
\]

**Differential approximation**

Step 1:

\[
y = f(x) = \sin x
\]
\[
\frac{dy}{dx} = f'(x) = \cos x
\]

\[
\frac{dy}{dx} = f'(x) \, dx = \cos x \, dx
\]

At \( x = \frac{\pi}{6} \):

\[
\frac{dy}{dx} = \cos \frac{\pi}{6} \, dx = \frac{\sqrt{3}}{2} \, dx
\]

\[
dx = 29^\circ - 30^\circ = -1^\circ = -\frac{\pi}{180}
\]

New old reference value

Then:

\[
\frac{dy}{dx} = \frac{\sqrt{3}}{2} \left( \frac{-\pi}{180} \right) = -0.01511
\]

Step 2:

At \( x = \frac{\pi}{6} \):

\[
y = f\left(\frac{\pi}{6}\right) = \sin \frac{\pi}{6} = \frac{1}{2}
\]

\[
= 0.50000
\]

New value of \( y = \) old value plus change:

\[
y + dy = 0.50000 - 0.01511
\]

\[
y = 0.48489 \quad \text{(done)}
\]

Compare with:

\[
\Delta y = \sin 29^\circ - \sin 30^\circ
\]

Change: \( \text{new } - \text{old }\)

\[
= -0.01511
\]

\[
dy = -0.01511
\]

The differential approximation is too high (less negative).

\[
\begin{align*}
\Delta y & \approx dy = dx \, c_0 \quad y = L(x) \\
& = f(x)
\end{align*}
\]

Remember, derivative formulas for trig functions assume angles are given in radians. Calculation must be done in radians.