Solving a functional relationship between 2 variables for the input variable (1)

$m = \frac{m_0}{\sqrt{1-v^2/c^2}}$ is a relationship (equation) between 2 physical variables m (output) and v (input)

$f(v) = \frac{m_0}{\sqrt{1-v^2/c^2}}$ is the definition of a function which produces a number (with units) from the input; f is the name of this function.

To find a formula for the inverse function, we must solve the equation in which v and m appear together for v. The expression we find for it will then define the formula for the inverse function.

f is the following sequence of operations:

input:	$v \geq 0$
square:	v^2
divide by c^2:	$\frac{v^2}{c^2}$
change sign:	$-\frac{v^2}{c^2}$
add 1:	$1-\frac{v^2}{c^2}$
take sqrt:	$\sqrt{1-\frac{v^2}{c^2}}$
take reciprocal:	$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$
mult by m_0:	$\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$
output:	$m = \frac{m_0}{\sqrt{1-v^2/c^2}}$

reverse process to go from output back to input: one-by-one apply inverse operations in reverse order

divide by m_0:	$\frac{m}{m_0} = \frac{1}{\sqrt{1-v^2/c^2}}$
take reciprocal:	$\frac{m_0}{m} = \sqrt{1-v^2/c^2}$
square:	$\left(\frac{m_0}{m}\right)^2 = 1-\frac{v^2}{c^2}$
subtract 1:	$\left(\frac{m_0}{m}\right)^2 - 1 = -\frac{v^2}{c^2}$
change sign:	$1-\left(\frac{m_0}{m}\right)^2 = \frac{v^2}{c^2}$
mult. by c^2:	$c^2\left(1-\left(\frac{m_0}{m}\right)^2\right) = v^2$
take square root:	$\sqrt{c^2\left(1-\left(\frac{m_0}{m}\right)^2\right)} = \sqrt{v^2} = v$ (since $v \geq 0$)
simplification steps:	$\sqrt{c^2}\sqrt{1-\left(\frac{m_0}{m}\right)^2} = c\sqrt{1-\frac{m_0^2}{m^2}}$

$\boxed{\text{you must do the same operation to both sides of the equation at each step}}$

final result:

$$\underbrace{v = c\sqrt{1-\frac{m_0^2}{m^2}}}_{\text{function of variable } m}$$

$$f^{-1}(m) = c\sqrt{1-\frac{m_0^2}{m^2}}$$

domain of f^{-1}

what are the allowed input values m?

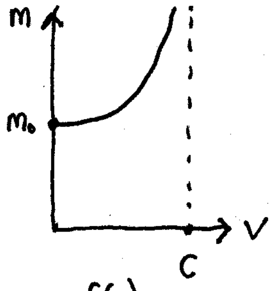
$$1-\frac{m_0^2}{m^2} \geq 0 \xrightarrow[\text{inequality}]{\text{solve}} 1 \geq \frac{m_0^2}{m^2} \rightarrow m^2 \geq m_0^2 \rightarrow \boxed{m \geq m_0}$$

same for domain of f:

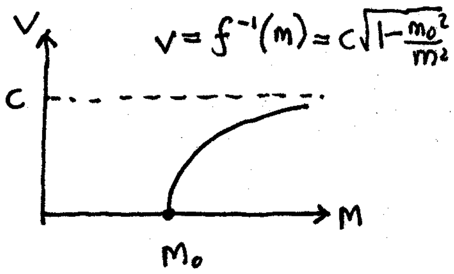
$$1-\frac{v^2}{c^2} > 0 \rightarrow 1 > \frac{v^2}{c^2} \rightarrow c^2 > v^2 \rightarrow c > v \quad \text{or} \quad v < c$$

since by assumption $v \geq 0$: $\boxed{0 \leq v < c}$

Graphs



$m = f(v)$
$= \dfrac{m_0}{\sqrt{1-v^2/c^2}}$



These are the same curves and same relationship between the two variables, but with the axes interchanged.
While the axes are still labeled v and m, it makes no sense to graph them together with these orientations since one then cannot use either variable name to label the axes.

The two constants c (with the same velocity units as the variable v, so that their ratio $\mathcal{V} = v/c$ is dimensionless) and m_0 (with the same mass units as the variable m, so that their quotient is dimensionless) set the scale for the two axes.

We could re-express these two equations in dimensionless form:

$$\frac{m}{m_0} = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \rightarrow M = \frac{1}{\sqrt{1-\mathcal{V}^2}} = F(\mathcal{V})$$

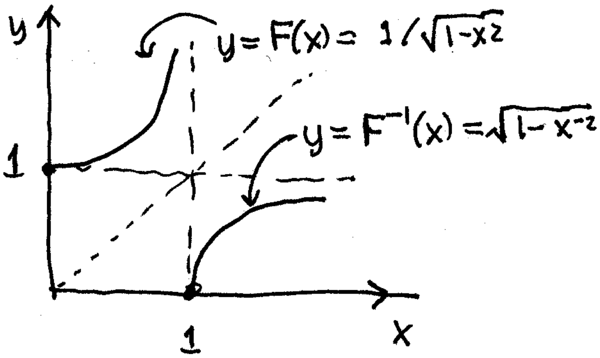
$$\frac{v}{c} = \sqrt{1-\left(\frac{m_0}{m}\right)^2} \rightarrow \mathcal{V} = \sqrt{1-1/M^2} = F^{-1}(M)$$

This corresponds to measuring the variables by multiples of these 2 constants:

"*Half the speed of light*" : $v = \frac{1}{2}c \rightarrow \mathcal{V} = \frac{1}{2}$

"*twice the rest mass*" : $m = 2m_0 \rightarrow M = 2$.

We can study the dimensionless mathematical functions with the default names x and y for inputs and outputs and graph them on the same axes:



Now we dont have to worry about conflicting interpretations of variable names. x is just the input, y the output.

[By introducing different variable names x and y we could even make such a graph for the original functions.]