functions of one independent variable : vocabulary

math is case-sensitive : lowercase and uppercase letters are considered distinct
different symbols : $r \neq R$, $t \neq T$, ...

single letters stand for (real or complex) numbers :

$x, X, y, Y, z, Z, t, T, a, A, b, B, \alpha, \beta, \gamma, \theta, \varphi, \Theta, \ldots$

usual choices (default choices for abstract math problems) (but not always)

end of alphabet	$x, y, z, u, v, t, \ldots$	"variable names" understood to take variable (changing) values
beginning	$a, b, c, A, B, C, \ldots$	"parameter names" standing for "constants"
almost middle	$f, g, h, F, G, \ldots$	function names

In applications these conventions are overridden by symbols traditionally used to represent physical quantities, like $A = s^2$ (area of a square is the square of the side length) or $A = \pi r^2$ (area of a circle is π times the square of the radius). Usually the first letter of the name of the quantity is used as its symbol. And standard functions have instead abbreviations: cos, sin, ..., arccos, arcsin, ..., exp, ln, ...

GOAL The primary activity in a calc course is "solving problems."

- worked example problems outline how one goes about solving a problem but does not always include every step (you should be able to fill in the missing steps)
- exercise problems: half of these (odd numbers) have the answer "in the back of the book" (or online homework answers!)

The goal of calc is not to get the answer in the back of the book. These answers are instead a means to an end, not the end itself. They tell you if you understood the process well enough to get through to the correct answer. Understanding the process is the real end goal here.

"Learning calculus" is about understanding its concepts and notation and how its various techniques can be used to understand the behavior of relationships between continuously changable variables. Part of learning calculus is understanding its terminology and notation and having some idea where its various rules and recipes come from (even though you don't have to remember the derivations after seeing them). There are thousands of calculus books, but they all share a common notation and vocabulary that enables us to communicate in an unambiguous language.

x^p+1 <u>expression</u> , x and p stand for real numbers, the expression is math shorthand for computing a number from given values of x and p called <u>evaluating the expression</u>.

$\underbrace{A}_{\text{LHS}} = \underbrace{B}_{\text{RHS}}$ <u>equation</u>, where A and B are both expressions standing for numbers.

$$\underbrace{f}_{\substack{\text{name or}\\ \text{symbol for}\\ \text{function}}}(\underbrace{x}_{\substack{\text{input}\\ \text{variable}}}) = \underbrace{x^{\overset{\text{parameter}}{p}}+1}_{\substack{\text{output}\\ \text{expression}}}$$

<u>function definition</u>, tells you how to <u>evaluate the function</u> with a given input number in place of the input variable.

$f(t) = t^p + 1$

The <u>input variable</u> is just a placeholder and the function definition does not care what name it has.

The letter p here is interpreted as a <u>parameter</u>, for each value of p we obtain a different function.
When a function definition contains a parameter or parameters, it defines a <u>family of functions</u>.

$f(1) = 1^p + 1$
$\quad\;\; = 1$

Replacing the input variable by a specific number (on both sides of the equation), we can evaluate the function to a number. In this case the result does not depend on the parameter because of the special properties of the number 1.

$f(2) = 2^p + 1$

In this case the result still depends on the parameter.
If $p = 1$, then $f(2) = 2^1 + 1 = 2 + 1 = 3$.

no : "when $x = 2$, $f(x) = 2^p + 1$ "
yes : $f(2) = 2^p + 1$

function notation allows us to be more efficient in expressing this statement

$x^2 + y^2 = xy + 1$

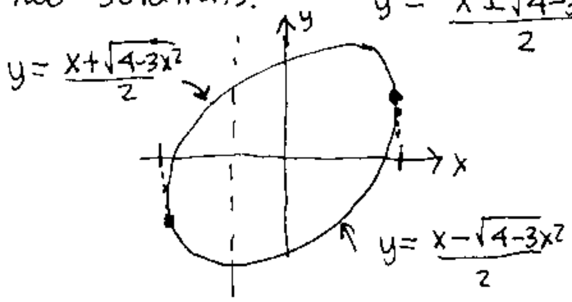
<u>relationship</u> between two otherwise independent variables x and y (two variables are "independent" when you can specify the value of one without affecting the value of the other). such a relationship makes one variable <u>depend</u> on the other and vice versa.

$$\underset{\substack{\uparrow\\ \text{dependent}\\ \text{variable}}}{y} = \underset{\substack{\uparrow\\ \text{independent}\\ \text{variable}}}{x}{}^p + 1$$

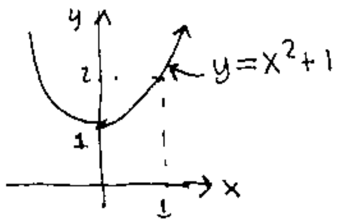
<u>functional relationship</u> (explicit!)
expresses one variable y in terms of a function of another variable x. The input variable is called the independent variable and the output variable the dependent variable.

functions of one independent variable: vocabulary (3)

curves in the plane: $x^2+y^2=xy+1$ and $y=x^2+1$ (setting $p=2$) can be visualized as curves in the xy plane. We can actually solve the first equation for y as a function of x using the quadratic formula, but there are two solutions: $y=\dfrac{x\pm\sqrt{4-3x^2}}{2}$ $\quad -\dfrac{2}{\sqrt{3}}\le x\le\dfrac{2}{\sqrt{3}}$ (so that $4-3x^2\ge 0$ to produce real numbers)



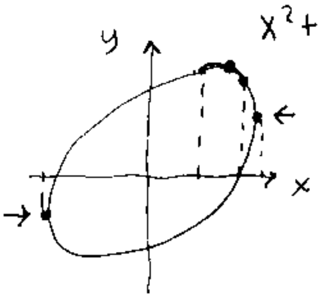
For a general relationship there may be more than one y value for each allowed x value and is not a function graph which has exactly one y value for each allowed x value (passes the vertical line test).



However, in this case our relationship curve is really two function graphs glued together.

implicit functional relationship, implicit means indirect

$x^2+y^2=xy+1$



for any point except the endpoints in this example, the relationship can be said to locally determine y as a function of x, that is a little piece of the curve on either side of the point belongs to a function graph, so indirectly (or implicitly) we can think of y as some unknown function of x (unless we can actually solve the relationship as in this example) near that point. (at the endpoints this can only be done on one side).

An explicit (direct) functional relationship tells us exactly what the function is.

$y=x^2+1$ and $s=t^2+1$ express the same functional relationship but in applications the variable names used represent physical variable values in some system of units so these variable names are hardwired into the problem and there is no function name (like $f(x)=x^2+1$) that is specifically mentioned.

functions of one independent variable : vocabulary (4)

explicit differentiation : we differentiate an explicit expression of the independent variable using derivative rules

$$f(x) = x^p + 1 \qquad\qquad y = x^p + 1$$

$$f'(x) = \underbrace{\frac{d}{dx}}\left(x^p+1\right) = p x^{p-1} \quad \text{or} \quad \frac{dy}{dx} = \frac{d}{dx}\left(x^p+1\right) = p x^{p-1}$$

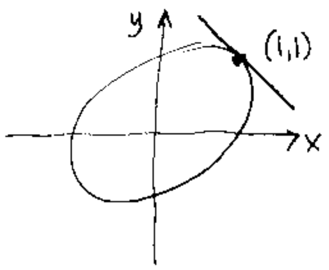
"take the derivative wrt x of the expression to the immediate right"

implicit differentiation : we have no explicit function to differentiate, only an equation relating two variables. We can choose either variable to be the "independent" variable and the other one will then be the "dependent variable" since it will depend on the first variable through this relationship.

$$x^2 + y^2 = xy + 1$$

$$\underbrace{1^2+1^2}_{2} = \underbrace{1\cdot 1+1}_{2} \rightarrow (\overset{x}{1},\overset{y}{1}) \text{ is a pt on the curve.}$$

$\left.\dfrac{dy}{dx}\right|_{\substack{x=1\\y=1}}$ is the slope of the tangent line there.



The rate of change of y wrt x can only be calculated indirectly as a derivative by differentiating the equation relating x and y :

$$\frac{d}{dx}\left[x^2+y^2 = xy+1\right]$$

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}\left(xy+1\right)$$

we need the chain rule :

$$\frac{d}{dx}F(x) = F'(x)$$

$$\frac{d}{dx}G(y) = G'(y)\frac{dy}{dx}$$

$$\underbrace{\frac{d}{dx}x^2}_{2x} + \underbrace{\frac{d}{dx}y^2}_{2y\frac{dy}{dx}} = \underbrace{\frac{d}{dx}(xy)}_{\underbrace{\frac{d(x)}{dx}}_{1}y + x\underbrace{\frac{d}{dx}y}_{\frac{dy}{dx}}} + \underbrace{\frac{d}{dx}(1)}_{0}$$

$$2x + 2y\frac{dy}{dx} = y + x\frac{dy}{dx}$$

now we have a relationship between x, y, and $\frac{dy}{dx}$

It is easily solved for $\frac{dy}{dx}$:

$$2y\frac{dy}{dx} - x\frac{dy}{dx} = y - 2x$$

$$(2y-x)\frac{dy}{dx} = y - 2x \rightarrow \boxed{\frac{dy}{dx} = \frac{y-2x}{2y-x}}$$

slope at a general pt (x,y) on the curve.

$$\left.\frac{dy}{dx}\right|_{\substack{x=1\\y=1}} = \frac{1-2(1)}{2(1)-1} = -1 .$$

Done.