On the three-dimensional spaces which admit a continuous group of motions[1,2]

Essay by member Luigi Bianchi

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[1]Original title: Sugli spazi a tre dimensioni che ammettono un gruppo continuo di movimenti, *Memorie di Matematica e di Fisica della Societa Italiana delle Scienze, Serie Terza*, **Tomo XI**, pp. 267–352 (1898). Printed with the kind permission of the Accademia Nazionale delle Scienze, detta dei XL, in Rome, the current copyright owner. Translated by Robert Jantzen, Department of Mathematical Sciences, Villanova University, Villanova, Pa 19085, USA.

[2]This paper was also reprinted in: *Opere* [The Collected Works of Luigi Bianchi], Rome, Edizione Cremonese, 1952, vol. **9**, pp. 17-109.

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Preface.

We define the metric of a space of n dimensions in the manner of Riemann by giving the expression for the square of its line element:

$$ds^2 = \sum_{i,k}^{1\ldots n} a_{ik}\, dx_i\, dx_k \ , \tag{1}$$

namely the law by which we measure infinitesimal arclengths in the space S_n, from which the law of measure for finite arclengths follows.

We consider n independent real variables $x_1, x_2, \ldots, x_n$ and assume that the coefficients a_{ik} of the quadratic differential form (1) as well as their first and second derivatives are real, finite and continuous functions of the x for the entire range of values which we consider. We also assume that the discriminant of expression (1) is always nonzero and that the coefficients a_{ik} fulfill the well known inequalities which make this differential form *positive-definite*.

It is well known how the law for measuring angles and the entire geometry of the space S_n is determined by equation (1). If two spaces S_n, S'_n can be put into a one-to-one correspondence in such a way that the line elements are the same, the two spaces will be called *isometric* and the two geometries will be identical. When the line elements of the two spaces only differ by a constant factor or can be reduced to this relationship by a transformation of coordinates, the two spaces will be called *similar*, and we will consider them as belonging to the same type. Their geometries are essentially identical; the only thing which changes from one to the other is the unit of linear measure.

An isometry of a space S_n into itself will be called a *motion* of this space. We will consider the spaces which admit *continuous* motions into themselves, namely, such that in the corresponding equations of the transformation appear some arbitrary parameters. The set of all these motions for a given S_n clearly forms a *group*. Simple geometrical considerations show that the number of parameters of this group is necessarily finite, which is in fact easily demonstrated analytically as we will see. If r is the number of these parameters in the complete group of motions, in every case this group will consist of a *finite-dimensional continuous* Lie[5] group G_r generated by r infinitesimal transformations $X_1 f, X_2 f, \ldots, X_r f$.

The problem of determining which spaces possess a continuous group of motions reduces therefore essentially to the classification of all possible forms of ds^2 which possess a Lie group $G_r \equiv (X_1 f, \ldots, X_n f)$ which transforms ds^2 into itself.

While the fundamental equations for the solution of this problem are already known from the work of Lie himself and of Killing, the problem has not been treated in complete form as far as I know. Indeed for arbitrary n, attention has been limited to the case in which figures in the space S_n can be transported with the *maximum* number of degrees of freedom: then the space is of constant curvature and the group possesses $r = n(n+1)/2$

[5] S. Lie-F. Engel, *Theorie der Transformationsgruppen*, Vol. I (1888), Chap. 18, p. 310 and Vol. III (1893), p. 575. [In the bibliographical footnotes, authors' first initials and, wherever missing, authors' names, have been added by the Editor. Also, journal titles were corrected and details of the citations were added wherever necessary. (Editor)]

parameters. Only for $n = 2$, namely for ordinary surfaces, do we know the complete solution of the problem, and it is known that the number of parameters can only fall into the two cases $r = 1$, $r = 3$. The surfaces of the first family are the one and only ones which are isometric to a surface of revolution; those of the second are exactly the surfaces of constant curvature.

In the present work I propose to study completely the case $n = 3$, in other words to classify all types of 3-dimensional spaces in which it is possible to transport figures along a certain degree of freedom. Apart from the extreme case of spaces of constant curvature which have a group G_6 of motions, there exist, as we will show, many intermediary types for which the number of parameters of the group can assume one of the four values $r = 1, 2, 3, 4$, while there do not exist spaces with groups of motions (or with partial subgroups) of 5 parameters.

To point out the main difference between the case of the surfaces $n = 2$ and that of $n = 3$, we remind ourselves that a surface which admits a transitive group of motions is necessarily of constant curvature, namely, if a point can be transported anywhere, it can also be rotated around every point. On the other hand there exist spaces of 3 dimensions in which we can transport any point of the space everywhere with a transformation, but the space is not of constant curvature; these spaces admit a transitive group of transformations with 3 or 4 parameters. In the spaces which admit only a group G_3 the entire space is fixed if we fix a single point. In the ones which admit a group G_4, it is still possible to have a continuous rotation G_1 around any arbitrary point P; however, together with P all the points of a certain geodesic through P remain fixed, so that these groups G_4 belong, according to the nomenclature of Lie, to the class of *systatic* groups. The space is then lined with a double infinity of such geodetic axes which completely fill the space, and besides the transformations (translations) which permit a point of a figure to be transported everywhere, there are still arbitrary rotations possible around any of these axes. Moreover, spaces which admit a group G_3 and those admitting a group G_4 are further distinguished into different irreducible types as we will see.

In the treatise of this problem I present here, I have constantly made use of the theorems and notations contained in the great work of Lie and more particularly his results on the composition of groups. They allow us to completely solve the question which approached directly would present great difficulties. Naturally the same method could be applied to a space of a larger number of dimensions, but as soon as $n > 3$, the investigation seems to get complicated very quickly.

1 The Killing Equations.[6]

Given a quadratic differential form in n variables:

$$ds^2 = \sum_{i,k}^{1..n} a_{ik}\, dx_i\, dx_k \ , \tag{1}$$

[6]W. Killing, *Über die Grundlagen der Geometrie*, Journ. für die r. und ang. Math. (Crelle), 109 (1892), 121–186.

we look for the conditions which this form must satisfy in order to admit the group G_1 generated by the infinitesimal transformation $Xf = \sum_r^{1..n} \xi_r \, \partial f / \partial x_r$.

It will therefore be *necessary and sufficient* that the operation Xf acting on the form (1) give an identically null result. Now we have:

$$X(ds^2) = \sum_{i,k} X(a_{ik})\, dx_i\, dx_k + \sum_{r,k} a_{rk}\, dX(x_r)\, dx_k + \sum_{i,r} a_{ir}\, dX(x_r)\, dx_i \ ,$$

namely

$$\begin{aligned} X(ds^2) \quad &= \quad \sum_{i,k,r} \xi_r \frac{\partial a_{ik}}{\partial x_r}\, dx_i\, dx_k + \sum_{r,k} a_{rk}\, d\xi_r\, dx_k + \sum_{i,r} a_{ir}\, d\xi_r\, dx_i \\ &= \quad \sum_{i,k} \left\{ \sum_r \left(\xi_r \frac{\partial a_{ik}}{\partial x_r} + a_{rk} \frac{\partial \xi_r}{\partial x_i} + a_{ir} \frac{\partial \xi_r}{\partial x_k} \right) \right\} dx_i\, dx_k \ . \end{aligned}$$

The n functions $\xi_1, \xi_2, \ldots, \xi_n$ therefore will have to fulfill the $n(n+1)/2$ linear homogeneous first order partial differential equations:

$$\sum_r \left\{ \xi_r \frac{\partial a_{ik}}{\partial x_r} + a_{ir} \frac{\partial \xi_r}{\partial x_k} + a_{kr} \frac{\partial \xi_r}{\partial x_i} \right\} = 0 \ , \qquad i, k = 1, 2, 3, \ldots, n \ . \tag{A}$$

Because the determinant of the a_{ik} is different from zero, these linear homogeneous equations in ξ and their first derivatives are linearly independent; moreover it is immediately seen that they are also linearly independent with respect to the n^2 first derivatives of ξ so that they can be solved for $n(n+1)/2$ of these derivatives, chosen conveniently. It is important to observe with Killing (*ibid.*, p.168) that by again differentiating the fundamental equations (A) all the second derivatives of the ξ can be obtained expressed linearly and homogeneously as functions of the first derivatives and the ξ themselves. In fact, we differentiate (A) with respect to x_l , obtaining:

$$\sum_r \left\{ \frac{\partial^2 a_{ik}}{\partial x_r \partial x_l} \xi_r + \frac{\partial a_{ik}}{\partial x_r} \frac{\partial \xi_r}{\partial x_l} + \frac{\partial a_{ir}}{\partial x_l} \frac{\partial \xi_r}{\partial x_k} + \frac{\partial a_{kr}}{\partial x_l} \frac{\partial \xi_l}{\partial x_i} \right.$$
$$\left. + a_{ir} \frac{\partial^2 \xi_r}{\partial x_k \partial x_l} + a_{kr} \frac{\partial^2 \xi_r}{\partial x_i \partial x_l} \right\} = 0 \ .$$

We then write the equations obtained from this last one by first interchanging k with l , then i with k , namely:

$$\sum_r \left\{ \frac{\partial^2 a_{il}}{\partial x_r \partial x_k} \xi_r + \frac{\partial a_{il}}{\partial x_r} \frac{\partial \xi_r}{\partial x_k} + \frac{\partial a_{ir}}{\partial x_k} \frac{\partial \xi_k}{\partial x_l} + \frac{\partial a_{rl}}{\partial x_k} \frac{\partial \xi_r}{\partial x_i} \right.$$
$$\left. + a_{ir} \frac{\partial^2 \xi_r}{\partial x_k \partial x_l} + a_{lr} \frac{\partial^2 \xi_r}{\partial x_i \partial x_k} \right\} = 0 \ ,$$

$$\sum_r \left\{ \frac{\partial^2 a_{kl}}{\partial x_r \partial x_i} \xi_r + \frac{\partial a_{kl}}{\partial x_r} \frac{\partial \xi_r}{\partial x_i} + \frac{\partial a_{kr}}{\partial x_i} \frac{\partial \xi_r}{\partial x_l} + \frac{\partial a_{rl}}{\partial x_i} \frac{\partial \xi_r}{\partial x_k} \right.$$
$$\left. + a_{kr} \frac{\partial^2 \xi_r}{\partial x_i \partial x_l} + a_{lr} \frac{\partial^2 \xi_r}{\partial x_i \partial x_k} \right\} = 0 \ .$$

Subtracting the first from the sum of these last two and dividing the result by 2, we obtain:

$$\sum_r \left\{ a_{rl} \frac{\partial^2 \xi_r}{\partial x_i \partial x_k} + \frac{\partial}{\partial x_r} [l, ik] \, \xi_r \right.$$
$$\left. + [r, ik] \frac{\partial \xi_r}{\partial x_l} + [l, ir] \frac{\partial \xi_r}{\partial x_k} + [l, kr] \frac{\partial \xi_r}{\partial x_i} \right\} = 0 \ ,$$
$$l, i, k = 1, 2, 3, \ldots, n \ , \tag{2}$$

where the Christoffel symbol $[l, ik]$ has the well known meaning

$$[l, ik] = \frac{1}{2} \left(\frac{\partial a_{il}}{\partial x_k} + \frac{\partial a_{kl}}{\partial x_i} - \frac{\partial a_{ik}}{\partial x_l} \right) \ .$$

If in (2) we fix i, k and let l take all the values from 1 to n, the equations thus obtained, since the determinant of a_{ik} is nonzero, can be solved for the second derivatives of ξ. To write down the solution we indicate by A_{ik} the adjoint of a_{ik} divided by the latter's determinant.[7] Multiplying (2) by A_{lv} and summing from $l = 1$ to $l = n$ we thus obtain the required equations:

$$\frac{\partial^2 \xi_v}{\partial x_i \partial x_k} + \sum_{r,l} A_{lv} \xi_r \frac{\partial}{\partial x_r} [l, ik] + \sum_{r,l} A_{lv} [r, ik] \frac{\partial \xi_r}{\partial x_l}$$
$$+ \sum_r \left\{ \begin{matrix} v \\ ir \end{matrix} \right\} \frac{\partial \xi_r}{\partial x_k} + \sum_r [v, kr] \frac{\partial \xi_r}{\partial x_i} = 0 \tag{B}$$
$$(i, k = 1, 2, 3, \ldots, n) \ ,$$

where the Christoffel symbol of the second kind $\left\{ \begin{smallmatrix} v \\ ir \end{smallmatrix} \right\}$ has the meaning

$$\left\{ \begin{matrix} v \\ ir \end{matrix} \right\} = \sum_k A_{kv} [k, ir] \ .$$

Equations (B) show us that the general integral of (A) contains the maximum number of arbitrary constants. In fact assuming the $n(n+1)/2$ functions

$$\xi_r \ , \frac{\partial \xi}{\partial x_i} \ , \qquad (i, r = 1, 2, \ldots, n)$$

as unknowns, using (B) we can express all their first derivatives as (linear and homogeneous) functions of the same unknowns and we therefore have a system of linear homogeneous total differential equations, the unknowns then being related by the $n(n+1)/2$ equations (A). The *maximum* number of arbitrary constants that can appear in the general integral of (A) will therefore be given by:[8]

$$r = n(n+1) - n(n+1)/2 = n(n+1)/2 \ .$$

If this maximal number is reached we will have the case of *complete* integrability and the space S_n, as is well known, will then be of constant curvature. In each case, the number r of independent infinitesimal transformations that the differential form (1) admits will be a finite number $r \leq n(n+1)/2$, and these r transformations $X_1 f, X_2 f, \ldots, X_r f$ will generate the continuous group G_r of motions of the space S_n.

[7] Namely, the inverse [Translator].

[8] The integral system is in fact specified if we give at one point of space the initial values of the $n(n+1)$ unknown functions which are, however, constrained by $n(n+1)/2$ independent relations.

2 Spaces which admit a group G_1.

From equations (A) we immediately deduce a consequence which is important to note; we can state: *two infinitesimal transformations of the space S_n cannot have common trajectories without coinciding.* And indeed we show immediately that if $\xi_1, \xi_2, \ldots, \xi_n$ satisfy equations (A) and $\lambda\xi_1, \lambda\xi_2, \ldots, \lambda\xi_n$ is a new set of solutions, the factor λ must necessarily be constant. In fact replacing ξ_r by $\lambda\xi_r$ in (A) gives

$$\sum_r \left(a_{ir}\xi_r \frac{\partial\lambda}{\partial x_k} + a_{kr}\xi_r \frac{\partial\lambda}{\partial x_i} \right) = 0 \ , \tag{3}$$

from which, setting $i = k$:

$$\sum_r a_{rs}\xi_r \frac{\partial\lambda}{\partial x_i} = 0 \ .$$

Assuming that $\partial\lambda/\partial x_s \neq 0$, it follows that $\sum_r a_{rs}\xi_r = 0$, and from (3), setting $k = s$, we deduce that $\sum_r a_{rs}\xi_r = 0$; but the determinant of the a is nonzero and this will imply that all the ξ are zero.

We now assume that the space S_n admits a group of motions G_1 generated by the infinitesimal transformation $Xf = \sum_i \xi_i \partial f/\partial x_i$. We can simplify the computations by assuming the trajectories of the group as the coordinate lines (x_1), so that we have $\xi_2 = \xi_3 = \cdots \xi_n = 0$, and by changing the parameters conveniently we can make $\xi_1 = 1$, namely $Xf = \partial f/\partial x_1$.[9]

Then (A) gives us simply

$$\frac{\partial a_{ik}}{\partial x_1} = 0 \qquad (i, k = 1, 2, \ldots, n) \ ,$$

which shows that the coefficients a_{ik} are independent of x_1. Viceversa it is clear that if in (1) the coefficients a_{ik} do not depend on x_1, the transformation $x_1' = x_1 + \textit{constant}$ gives a continuous group G_1 of transformations in the space. And as long as the a_{ik} remain arbitrary functions of the other variables $x_2, x_3, \ldots, x_n$, this group G will be the *complete* group of motions.

In the case $n = 2$ we then recover the well known result that the surface is isometric to a surface of rotation.

3 Surfaces with a group G_2.

We now study the types of ds^2 which admit a group G_2 of motions, assuming that the number of variables is $n = 2$ or $n = 3$. The result for $n = 2$ is well known but it is worthwhile to rederive it again here.

So let us first assume $n = 2$ and indicate by $X_1 f$, $X_2 f$ the two infinitesimal transformation generators of the group G_2 under consideration. Replacing $X_1 f$, $X_2 f$ by new convenient linear combinations of them, we can always reduce ourselves to the case in which we have for the composition equations[10]

$$(a) \qquad [X_1, X_2]f = 0 \ , \ \text{or} \qquad (b) \qquad [X_1, X_2]f = X_1 f \ .$$

[9] It is sufficient to assume as new variables $y_1, y_2, \ldots, y_n$ an integral of the equation $X(y_1) = 1$ and $n-1$ independent integrals of the equation $X(y) = 0$.

[10] S. Lie-F. Engel, Vol. III, p. 713

The trajectories of the two infinitesimal transformation generators being in each case distinct (§2), we can assume them respectively as coordinate lines and we then have $X_1 f = \xi\, \partial f/\partial x_1$, $X_2 f = \eta\, \partial f/\partial x_2$. In case ($a$) it follows that $\partial\xi/\partial x_2 = 0$, $\partial\eta/\partial x_1 = 0$, so that by making a change of parameters, we can assume $\xi = \eta = 1$.

Since the equations (A) have to be satisfied either with $\xi_1 = 1$, $\xi_2 = 0$ or with $\xi_1 = 0$, $\xi_2 = 1$, it follows that the coefficients of the differential form

$$ds^2 = a_{11}\, dx_1{}^2 + 2a_{12}\, dx_1 dx_2 + a_{22}\, dx_2{}^2$$

are constants and with a (linear) change of variables we can therefore have $ds^2 = dx_1{}^2 + dx_2{}^2$, hence the surface has zero curvature. The complete group of motions is the G_3 generated by the three infinitesimal transformations

$$X_1 f = \frac{\partial f}{\partial x_1}\ ,\ X_2 f = \frac{\partial f}{\partial x_2}\ ,\ X_3 f = x_2 \frac{\partial f}{\partial x_1} - x_1 \frac{\partial f}{\partial x_2}\ .$$

In case (b) we must have $\partial\eta/\partial x_1 = 0$, $-\eta\, \partial\xi/\partial x_2 = \xi$, and by changing the parameters x_1, x_2, we can set $\eta = 1$, $\xi = e^{-x_2}$, so that

$$X_1 f = e^{-x_2}\, \partial f/\partial x_1\ ,\ X_2 f = \partial f/\partial x_2\ .$$

Equations (A), assuming successively $\xi_1 = 0$, $\xi_2 = 1$ and $\xi_1 = e^{-x_2}$, $\xi_2 = 0$, give us

$$\frac{\partial a_{11}}{\partial x_2} = \frac{\partial a_{12}}{\partial x_2} = \frac{\partial a_{22}}{\partial x_2} = 0\ ,$$
$$\frac{\partial a_{11}}{\partial x_1} = 0\ ,\ \frac{\partial a_{12}}{\partial x_1} = a_{11}\ ,\ \frac{\partial a_{22}}{\partial x_1} = 2a_{12}\ ,$$

from which by integration we have $a_{11} = \alpha$, $a_{12} = \alpha x_1 + \beta$, $a_{22} = \alpha x_1{}^2 + 2\beta x_1 + \gamma$, with α, β, γ constants. Without loss of generality we can assume $\alpha = 1$ (by absorbing it into x_1), and writing x_1 in place of $x_1 + \beta$, we will have

$$ds^2 = dx_1^2 + 2x_1\, dx_1 dx_2 + (x_1{}^2 + R^2)\, dx_2{}^2\ .$$

If we set $x_1 = -ve^{u/R}$, $x_2 = u/R$, we obtain the typical (parabolic) form

$$ds^2 = du^2 + e^{2u/R} dv^2$$

of the line element of the pseudo-spherical surface. The complete group of motions is the G_3 generated by the infinitesimal transformations:

$$X_1 f = e^{-x_2} \frac{\partial f}{\partial x_1}\ ,\ X_2 f = \frac{\partial f}{\partial x_2}\ ,$$
$$X_3 f = \frac{1}{2} e^{x_2} (x_1{}^2 + R^2) \frac{\partial f}{\partial x_1} - x_1 e^{x_2} \frac{\partial f}{\partial x_2}\ .$$

The subgroup G_2 under consideration consists of all those groups G_1 which have as trajectories the geodetic circles (with ideal center) inclined at a constant angle to the parallel oricycles[11] $x_2 = constant$.[12]

In the analysis of the present § only the surfaces of constant zero or negative curvature have appeared, not those of constant positive curvature. The reason for this is the fact that the latter surfaces admit a group G_3 of motions, but never a *real* 2-parameter subgroup.

[11]In Italian: "oricicli" [Translator].

[12]If one represents these surfaces as pseudo-spheres these trajectories are loxodromes of the surfaces.

4 Spaces of 3 dimensions with a group G_2.

We now turn our attention to 3-dimensional spaces which admit a 2-parameter group of motions. The trajectories of the two infinitesimal transformation generators of this G_2 being distinct (§2), each point of the space will be moved over a surface by the transformations of G_2. We have therefore a family of surfaces Σ which represent for our group what Lie calls the *minimum invariant varieties*. For a given transformation of the G_2, any one of the Σ is transformed into itself and consequently any surface geodesically parallel to a Σ as well; we deduce from this that the ∞^1 surfaces Σ are geodesically parallel;[13] moreover, any each of them, admitting a group G_2 of transformations, will be of constant zero or negative curvature (§3). If we take the surfaces Σ as coordinate surfaces $x_1 = constant$ and their orthogonal trajectories for coordinate lines[14] x_1, we put the line element into the geodetic form:

$$ds^2 = dx_1{}^2 + a_{22}\, dx_2{}^2 + 2a_{23}\, dx_2 dx_3 + a_{33}\, dx_3{}^2 \ . \tag{4}$$

In each of the infinitesimal transformations $X_1 f$, $X_2 f$, since $\xi_1 = 0$, the equations (A), setting $i = 1$, $k = 2, 3$, give

$$a_{22}\frac{\partial \xi_2}{\partial x_1} + a_{23}\frac{\partial \xi_3}{\partial x_1} = 0 \ , \ a_{23}\frac{\partial \xi_2}{\partial x_1} + a_{33}\frac{\partial \xi_3}{\partial x_1} = 0 \ ,$$

from which since $a_{22}a_{33} - a_{23}{}^2 \neq 0$, we conclude that $\partial\xi_2/\partial x_1 = \partial\xi_3/\partial x_1 = 0$, namely that the coefficients of $X_1 f$, $X_2 f$ are independent of x_1.

Assuming this to be true, we take the respective (distinct) trajectories of $X_1 f$, $X_2 f$ as coordinate lines over one of the surfaces $x_1 = constant$ and we will have $X_1 f = \xi\, \partial f/\partial x_2$, $X_2 f = \eta\, \partial f/\partial x_3$.

We now distinguish again the two cases

$$(a) \qquad [X_1, X_2]f = 0 \qquad \text{and} \qquad (b) \qquad [X_1, X_2]f = X_2 f \ .$$

In the first case, as in the preceding §, we can make

$$X_1 f = \partial f/\partial x_2 \ , \ X_2 f = \partial f/\partial x_3$$

[13]We can deduce the same conclusion from the fundamental equations (A). Let us assume in fact Σ for the x_1 coordinate surfaces and for the x_1 coordinate lines [translator note: second x_1 corrected from Bianchi typo x_3 here and in the text as well] their orthogonal trajectories; we will have

$$ds^2 = a_{11}\, dx_1{}^2 + a_{22}\, dx_2{}^2 + 2a_{23}\, dx_2 dx_3 + a_{33}\, dx_3{}^2 \ .$$

If $X_1 f$, $X_2 f$ are their infinitesimal transformation generators, we have to have $X_1(x_1) = 0$, $X_2(x_1) = 0$ and consequently $X_1 f = \xi_2\, \partial f/\partial x_2 + \xi_3\, \partial f/\partial x_3$, $X_2 f = \eta_2\, \partial f/\partial x_2 + \eta_3\, \partial f/\partial x_3$. Now applying (A) successively to $X_1 f$, $X_2 f$ setting $i = k = 1$ we deduce

$$\frac{\partial a_{11}}{\partial x_2}\xi_2 + \frac{\partial a_{11}}{\partial x_3}\xi_3 = 0 \ , \ \frac{\partial a_{11}}{\partial x_2}\eta_2 + \frac{\partial a_{11}}{\partial x_3}\eta_3 = 0 \ ,$$

from which, since $\begin{vmatrix} \xi_2 & \xi_3 \\ \eta_2 & \eta_3 \end{vmatrix} \neq 0$ from §2, it follows that $\partial a_{11}/\partial x_2 = \partial a_{11}/\partial x_3 = 0$. Changing the parameter x_1, one can therefore make $a_{11} = 1$ which gives to the line element the geodetic form of the text.

[14]In the original text, "coordinate lines x_3", which is incorrect [Editor].

and the line element of the space will take the form

$$ds^2 = dx_1{}^2 + \alpha\, dx_2{}^2 + 2\beta\, dx_2 dx_3 + \gamma\, dx_3{}^2 \ , \qquad (a^*)$$

with α, β, γ being functions only of x_1.

In case (b) we take

$$X_1 f = \partial f/\partial x_3 \ , \ X_2 f = e^{x_3}\, \partial f/\partial x_2$$

and we will then have

$$ds^2 = dx_1{}^2 + \alpha\, dx_2{}^2 + 2(\beta - \alpha x_2)\, dx_2 dx_3 + (\alpha x_2{}^2 - 2\beta x_2 + \gamma)\, dx_3{}^2 \ , \qquad (b^*)$$

where α, β, γ are still functions only of x_1.

Vice versa, whatever are the functions α, β, γ of x_1 in (a^*) or (b^*), we will have a space which admits the 2-parameter group of motions $(\partial f/\partial x_2, \partial f/\partial x_3)$ in the first case and another $(\partial f/\partial x_3, e^{x_3}\partial f/\partial x_2)$ in the second case. If α, β, γ remain arbitrary functions of x_1, this G_2 is the complete group of motions, as will be shown by the analysis in the following §§.

5 Spaces with an intransitive group G_r of motions ($r \geq 3$).

We now pass to the treatment of 3-dimensional spaces which admit a group of motions with more than two parameters, beginning with the case in which this group G_r is intransitive.

From the considerations of the preceding § the minimum invariant varieties with respect to the group will be geodesically parallel surfaces, and because each of these has to admit a group G_r with $r \geq 3$ parameters,[15] one necessarily must have $r = 3$. To the line element of the space we then give the geodetic form

$$ds^2 = dx_1^2 + a_{22}\, dx_2^2 + 2a_{23}\, dx_2 dx_3 + a_{33}\, dx_3^2 \qquad (4)$$

and the geodesically parallel surfaces $x_1 = \mathit{constant}$ will be of constant curvature.

Arbitrarily selecting one of these, say $x_1 = 0$, we distinguish three cases characterized by the curvature K being zero, positive or negative. By substituting for this space a similar space, we can assume successively

$$K_0 = 0 \ , \ K_0 = 1 \ , \ K_0 = -1$$

and correspondingly we can change the coordinate lines of x_2, x_3 on the surface $x_1 = 0$ so that the line element ds_0^2 of $x_1 = 0$ assumes the respective typical forms:

$$K_0 = 0: \qquad ds_0^2 = dx_2^2 + dx_3^2 \ , \qquad (\alpha)$$
$$K_0 = 1: \qquad ds_0^2 = dx_2^2 + \sin^2 x_3\, dx_3^2 \ , \qquad (\beta)$$
$$K_0 = -1: \qquad ds_0^2 = dx_2^2 + e^{2x_3} dx_3^2 \ . \qquad (\gamma)$$

[15] That on any surface Σ the group G_r retains r parameters follows immediately from what we have seen in §4 because if we take the line element in the geodetic form (4), in every single infinitesimal transformation of G_r one has $\xi_1 = 0$ and ξ_2, ξ_3 are independent of x_1. Of course this is also clear geometrically since if all the points of a surface Σ remain fixed, the entire space is immobilized.

The group G_3 of motions of $x_1 = 0$ into itself will be generated respectively by the three infinitesimal transformations:

$$X_1 f = \frac{\partial f}{\partial x_2} \, , \qquad X_2 f = \frac{\partial f}{\partial x_3} \, , \; X_3 f = x_3 \frac{\partial f}{\partial x_2} - x_2 \frac{\partial f}{\partial x_3} \; ; \tag{α^*}$$

$$\begin{aligned} X_1 f = \frac{\partial f}{\partial x_3} \, , \qquad & X_2 f = \sin x_3 \frac{\partial f}{\partial x_2} + \cot x_2 \cos x_3 \frac{\partial f}{\partial x_3} \, , \\ & X_3 f = \cos x_3 \frac{\partial f}{\partial x_2} - \cot x_2 \sin x_3 \frac{\partial f}{\partial x_3} \; ; \end{aligned} \tag{β^*}$$

$$\begin{aligned} X_1 f = \frac{\partial f}{\partial x_3} \, , \qquad & X_2 f = \frac{\partial f}{\partial x_2} - x_3 \frac{\partial f}{\partial x_3} \, , \\ & X_3 f = x_3 \frac{\partial f}{\partial x_2} + \frac{1}{2}(e^{-2x_2} - x_3^2) \frac{\partial f}{\partial x_3} \; . \end{aligned} \tag{γ^*}$$

In all three cases these are also the infinitesimal transformations of the group of motions of the whole space. Now if for the line element (4) we write the three equations which result from the fundamental equations (A) setting $\xi_1 = 0$ and successively $(i, k) = (2, 2), (2, 3), (3, 3)$, we find

$$\begin{aligned} & \frac{\partial a_{22}}{\partial x_2}\xi_2 + \frac{\partial a_{22}}{\partial x_3}\xi_3 + 2a_{22}\frac{\partial \xi_2}{\partial x_2} + 2a_{23}\frac{\partial \xi_3}{\partial x_2} = 0 \, , \\ & \frac{\partial a_{23}}{\partial x_2}\xi_2 + \frac{\partial a_{23}}{\partial x_3}\xi_3 + a_{22}\frac{\partial \xi_2}{\partial x_3} + a_{23}\left(\frac{\partial \xi_2}{\partial x_2} + \frac{\partial \xi_3}{\partial x_3}\right) + a_{33}\frac{\partial \xi_3}{\partial x_2} = 0 \, , \\ & \frac{\partial a_{33}}{\partial x_2}\xi_2 + \frac{\partial a_{33}}{\partial x_3}\xi_3 + 2a_{23}\frac{\partial \xi_2}{\partial x_3} + 2a_{33}\frac{\partial \xi_3}{\partial x_3} = 0 \, . \end{aligned} \tag{C}$$

These must be satisfied when for ξ_2, ξ_3 in the three respective cases we substitute the three pairs of values which belong respectively to the 3 generator substitutions (α^*), (β^*) or (γ^*).

6 Discussion of the system (C).

We begin with case (α^*) and putting into (C) first[16] $\xi_2 = 1, \xi_3 = 0$ and then $\xi_2 = 0, \xi_3 = 1$, we deduce from this that $\partial a_{ik}/\partial x_2 = \partial a_{ik}/\partial x_3 = 0$ $(i, k = 2, 3)$, from which it follows that the coefficients a_{ik} here are functions only of x_1. If we now introduce into (C) the values $\xi_2 = x_3$, $\xi_3 = -x_2$ which belong to the third infinitesimal transformation, we have $a_{23} = 0$, $a_{22} = a_{33}$ and therefore for the line element of the space

$$ds^2 = dx_1^2 + \varphi^2(x_1)\,(dx_2^2 + dx_3^2) \, , \tag{5}$$

where $\varphi(x_1)$ indicates an (arbitrary) function of x_1.

In case (β), first setting in (C) $\xi_2 = 0$, $\xi_3 = 1$, the values which correspond to $X_1 f$, we see that a_{22}, a_{23}, a_{33} do not depend on x_3. Then substituting the values $\xi_2 = \sin x_3$, $\xi_3 = \cot x_2 \cos x_3$ corresponding to $X_3 f$, the first of (C) gives us $\sin x_3\, \partial a_{22}/\partial x_2 = 2\cos x_3 / \sin^2 x_2\, a_{23}$, and since neither a_{22} nor a_{23} depend on x_3, it follows that $a_{23} = 0$, $\partial a_{22}/\partial x_2 = 0$ and consequently $a_{22} = \varphi^2(x_1)$.

[16]The original paper had (ξ_1, ξ_2) instead of (ξ_2, ξ_3) here, which was an obvious typo [Editor].

The second of (C) then gives immediately $a_{33} = \sin^2 x_2\, \varphi^2(x_1)$, so that the line element of the space has the form

$$ds^2 = dx_1^2 + \varphi^2(x_1)\,(dx_2^2 + \sin^2 x_2\, dx_3^2)\ . \tag{6}$$

Finally in case (γ), equations (C) with the values $\xi_2 = 0$, $\xi_3 = 1$ belonging to $X_1 f$ show that a_{22}, a_{23}, a_{33} are again independent of x_3. Substituting next the values $\xi_2 = 1$, $\xi_3 = -x_3$ corresponding to $X_2 f$ we find[17]: $\partial a_{22}/\partial x_2 = 0$, $\partial a_{23}/\partial x_2 = a_{23}$, $\partial a_{33}/\partial x_2 = 2a_{33}$, and finally with the values $\xi_2 = x_3$, $\xi_3 = \frac{1}{2}(e^{-2x_2} - x_3^2)$ belonging to $X_3 f$: $a_{23} = 0$, $a_{22} = a_{33}\, e^{-2x_2}$, from which we arrive at the line element

$$ds^2 = dx_1^2 + \varphi^2(x_1)\,(dx_2^2 + e^{2x_2} dx_3^2)\ . \tag{7}$$

Vice versa for any function $\varphi(x_1)$ the spaces of the line elements (5), (6), (7) admit the respective intransitive group G_3 of motions (α^*), (β^*) or (γ^*).

We must now discover for which special forms of the function $\varphi(x_1)$ it will happen that the complete group of motions of the space will be larger.

7 The complete group of motions of the space: $ds^2 = dx_1^2 + \varphi^2(x_1)\,(dx_2^2 + dx_3^2)$.

In order to determine the most general infinitesimal motion $Xf = \eta_1\, \partial f/\partial x_1 + \eta_2\, \partial f/\partial x_2 + \eta_3 \partial f/\partial x_3$ of the present space, the fundamental equations (A), setting successively (i, k) = (1,1), (2,2), (3,3), (1,2), (1,3), (2,3) give the following 6 equations:[18]

$$\frac{\partial \eta_1}{\partial x_1} = 0\ , \tag{8}$$

$$\frac{\partial \eta_2}{\partial x_2} + \frac{\varphi'}{\varphi}\eta_1 = 0\ , \tag{9}$$

$$\frac{\partial \eta_3}{\partial x_3} + \frac{\varphi'}{\varphi}\eta_1 = 0\ , \tag{10}$$

$$\frac{\partial \eta_1}{\partial x_2} + \varphi^2(x_1)\frac{\partial \eta_2}{\partial x_1} = 0\ , \tag{11}$$

$$\frac{\partial \eta_1}{\partial x_3} + \varphi^2(x_1)\frac{\partial \eta_3}{\partial x_1} = 0\ , \tag{12}$$

$$\frac{\partial \eta_2}{\partial x_3} + \frac{\partial \eta_3}{\partial x_2} = 0\ . \tag{13}$$

By taking $\eta_1 = 0$ naturally one has only the three transfomations (α^*) and the question to be examined is therefore this: if the above equations can be satisfied with $\eta_1 \neq 0$.

Differentiating (9) with respect to x_1, (11) with respect x_2 and comparing, with the observation that by (8), η_1 does not depend on x_1, we find that

$$\frac{\partial^2 \eta_1}{\partial x_2^2} = (\varphi''\varphi - \varphi'^2)\,\eta_1\ , \tag{14}$$

[17] In the original paper, the third equation was $\partial a_{33}/\partial x_3 = 2a_{33}$, which was incorrect [Editor].

[18] The prime indicates the derivative with respect to x_1.

and similarly from (10), (12)

$$\frac{\partial^2\eta_1}{\partial x_3^2} = (\varphi''\varphi - \varphi'^2)\,\eta_1 \ . \tag{15}$$

Since η_1 is different from zero and does not depend on x_1, while φ is a function only of x_1, the resulting equations (14), (15) show that one will have:

$$\varphi''\varphi - \varphi'^2 = c \ , \tag{16}$$

$$\frac{\partial^2\eta_1}{\partial x_2^2} = \frac{\partial^2\eta_1}{\partial x_3^2} = c\eta_1 \ , \tag{17}$$

where c is a constant. Integrating (11), (12) we find

$$\begin{aligned} \eta_2 &= -\frac{\partial\eta_1}{\partial x_2}\int\frac{dx_1}{\varphi^2(x_1)} + \psi(x_2,x_3) \ , \\ \eta_3 &= -\frac{\partial\eta_1}{\partial x_3}\int\frac{dx_1}{\varphi^2(x_1)} + \chi(x_2,x_3) \ , \end{aligned} \tag{18}$$

where ψ, χ are two functions only of x_2, x_3. By substituting these into (13) it follows that

$$2\frac{\partial^2\eta_1}{\partial x_2\partial x_3}\int\frac{dx_1}{\varphi^2(x_1)} = \frac{\partial\psi}{\partial x_3} + \frac{\partial\psi}{\partial x_2} \ ,$$

from which, since η_1, ψ, χ are independent of x_1 while the integral necessarily contains it, we have

$$\frac{\partial^2\eta_1}{\partial x_2\partial x_3} = 0 \ . \tag{17*}$$

Comparing with (17), we have immediately $c\,\partial\eta_1/\partial x_2 = 0$, $c\,\partial\eta_1/\partial x_3 = 0$.

If $c \neq 0$ we will therefore have $\eta_1 = constant$, $\eta_2 = \psi(x_2,x_3)$, $\eta_3 = \chi(x_2,x_3)$, from which (9) or (10) shows that one has $\varphi'/\varphi = constant$. But this last result follows even if $c = 0$, since then by (17) and (17*), η_1 is a linear function of x_2, x_3 and since by (18) $\partial\eta_2/\partial x_2 = \partial\psi/\partial x_2$, (9) gives us: $\varphi'/\varphi = -1/\eta_1\,\partial\psi/\partial x_2$, from which we can conclude again that $\varphi'/\varphi = constant$. Therefore if the present space admits a larger group of motions (with $r > 3$ parameters) we necessarily have $\varphi' = k\varphi$ (k constant).

If $k = 0$ one can make $\varphi(x_1) = 1$ and have ordinary Euclidean space. If $k \neq 0$ one can assume that $\varphi(x_1) = e^{kx_1}$, and have the space of constant negative curvature $K = -k^2$.

In both cases the complete group of motions has 6 parameters. The result being well known, we do not concern ourselves with giving the actual 6 infinitesimal transformation generators, which are obtained by integrating the above equations.

8 The complete group of motions of the space: $ds^2 = dx_1^2 + \varphi^2(x_1)\,(dx_2^2 + \sin^2 x_2\,dx_3^2)$.

We proceed as in the previous §, writing first the equations which follow from (A) in order to find the most general infinitesimal motion of the space under consideration. We thus find

$$\frac{\partial\eta_1}{\partial x_1} = 0 \ , \tag{19}$$

$$\frac{\partial \eta_2}{\partial x_1} = -\frac{1}{\varphi^2}\frac{\partial \eta_1}{\partial x_2} \ , \tag{20a}$$

$$\frac{\partial \eta_2}{\partial x_2} = -\frac{\varphi'}{\varphi}\eta_1 \ , \tag{20b}$$

$$\frac{\partial \eta_3}{\partial x_1} = -\frac{1}{\varphi^2 \sin^2 x_2}\frac{\partial \eta_1}{\partial x_3} \ , \tag{21a}$$

$$\frac{\partial \eta_3}{\partial x_3} = -\frac{\varphi'}{\varphi}\eta_1 - \cot x_2\, \eta_2 \ , \tag{21b}$$

$$\frac{\partial \eta_2}{\partial x_3} + \sin^2 x_2 \frac{\partial \eta_3}{\partial x_2} = 0 \ . \tag{22}$$

Eliminating by differentiation η_2 from (20) and η_3 from (21), we find

$$\frac{\partial^2 \eta_1}{\partial x_2^2} = (\varphi''\varphi - \varphi'^2)\, \eta_1 \ ,$$
$$\frac{\partial^2 \eta_1}{\partial x_3^2} = (\varphi''\varphi - \varphi'^2) \sin^2 x_2\, \eta_1 - \sin x_2 \cos x_2 \frac{\partial \eta_1}{\partial x_2} \ ,$$

from which, since $\eta_1 \neq 0$ doesn't depend on x_1, we conclude that

$$\varphi''\varphi - \varphi'^2 = c \quad \text{(constant)} \ ,$$

$$\frac{\partial^2 \eta_1}{\partial x_2^2} = c\eta_1 \ , \ \frac{\partial^2 \eta_1}{\partial x_3^2} = c \sin^2 x_2\, \eta_1 - \sin x_2 \cos x_2 \frac{\partial \eta_1}{\partial x_2} \ . \tag{23}$$

Integrating the first of (20) and the first of (21) with respect to x_1 we have:

$$\eta_2 = -\frac{\partial \eta_1}{\partial x_2}\int \frac{dx_1}{\varphi^2(x_1)} + \psi(x_2, x_3) \ ,$$
$$\eta_3 = -\frac{1}{\sin^2 x_2}\frac{\partial \eta_1}{\partial x_3}\int \frac{dx_1}{\varphi^2(x_1)} + \chi(x_2, x_3) \ , \tag{24}$$

and substituting into (22) we obtain

$$2\left(\frac{\partial^2 \eta_1}{\partial x_2 \partial x_3} - \cot x_2 \frac{\partial \eta_1}{\partial x_3}\right)\int \frac{dx_1}{\varphi^2(x_1)} = \frac{\partial \psi}{\partial x_3} + \sin^2 x_2 \frac{\partial \chi}{\partial x_2} \ .$$

Since x_1 appears here only in the integrals, we necessarily have

$$\frac{\partial^2 \eta_1}{\partial x_2 \partial x_3} = \cot x_2 \frac{\partial \eta_1}{\partial x_3} \ ,$$

and if we differentiate this with respect to x_2 and the first of (23) with respect to x_3, we conclude that $(c+1)\, \partial\eta_1/\partial x_3 = 0$, and consequently $c = -1$ or $\partial\eta_1/\partial x_3 = 0$.

We consider in this § the first case $c = -1$; then from

$$\varphi''\varphi - \varphi'^2 = -1 \ , \tag{25}$$

it follows by differentiation that $\varphi''' - \varphi'\varphi'' = 0$, so that $\varphi'' = k\varphi$, (k constant) and (25) becomes $\varphi'^2 = 1 + k\varphi^2$.

If $k = 0$, neglecting the additive constant in x_1 we have $\varphi(x_1) = x_1$. If k is negative, we put $k = -1/R^2$ and we will have $\varphi(x_1) = R\sin(x_1/R)$; finally if k is positive, let $k = 1/R^2$ and it will be $\varphi(x_1) = R\sinh(x_1/R)$.

We have as a consequence the following three forms of the line element of the space:

$$\begin{aligned}
ds^2 &= dx_1^2 + x_1^2\,(dx_2^2 + \sin^2 x_2\, dx_3^2)\ ,\\
ds^2 &= dx_1^2 + R^2\sin^2(x_1/R)\,(dx_2^2 + \sin^2 x_2\, dx_3^2)\ ,\\
ds^2 &= dx_1^2 + R^2\sinh^2(x_1/R)\,(dx_2^2 + \sin^2 x_2\, dx_3^2)\ .
\end{aligned}$$

The first form belongs to ordinary Euclidean space (in polar coordinates), the second and third respectively to spaces of constant positive or negative curvature $K = \pm 1/R^2$. In all three cases the complete group of motions has 6 parameters.

9 The group G_3 of motions of the space: $ds^2 = dx_1^2 + dx_2^2 + \sin^2 x_2\, dx_3^2$.

In order to complete the discussion of the previous § there remains to be treated the case in which we have $\partial\eta_1/\partial x_3 = 0$. Equations (23) then become[19]

$$\frac{\partial^2\eta_1}{\partial x_2^2} = c\eta_1\ ,\ \frac{\partial\eta_1}{\partial x_2} = c\tan x_2\,\eta_1\ ,$$

from which by differentiating the second with respect to x_2 and comparing with the first we conclude (since by assumption $\eta_1 \neq 0$): $c(c+1) = 0$.

Since the case $c = -1$ has already been discussed in the previous §, there remains for us here only to assume $c = 0$ so that $\eta_1 = a$ (constant). Then (24) become $\eta_2 = \psi(x_2, x_3)$, $\eta_3 = \chi(x_2, x_3)$ and (20), (21), (22) give us

$$\frac{\partial\psi}{\partial x_2} + a\frac{\varphi'}{\varphi} = 0\ , \tag{26}$$

$$\frac{\partial\chi}{\partial x_3} + a\frac{\varphi'}{\varphi} + \cot x_2\,\psi = 0\ , \tag{27}$$

$$\frac{\partial\psi}{\partial x_3} + \sin^2 x_2\,\frac{\partial\chi}{\partial x_2} = 0\ . \tag{28}$$

In (26), (27) x_1 should appear only in φ'/φ and therefore $\varphi'/\varphi = k$ (constant), so that $\psi = -akx_2 + \theta(x_3)$, with θ a function only of x_3. After this (27), (28) become:

$$\frac{\partial\chi}{\partial x_2} = -\frac{\theta'(x_3)}{\sin^2 x_2}\ ,\ \frac{\partial\chi}{\partial x_3} = -ak + akx_2\cot x_2 - \cot x_2\,\theta(x_3)\ , \tag{29}$$

Forming the integrability condition for these last two equations, we conclude that $\theta''(x_3) + \theta(x_3) = ak(x_2 - \cos x_2 \sin x_2)$, so that $ak = 0$ and since $\eta_1 = a \neq 0$, we must have $k = 0$. So one therefore has $\varphi = \textit{constant}$ and without loss of generality (by substituting a similar space), we can make $\varphi(x_1) = 1$, which gives us the line element

$$ds^2 = dx_1^2 + dx_2^2 + \sin^2 x_2\, dx_3^2$$

[19]In the second equation, the original paper has a second derivative, which is incorrect [Editor].

indicated in the title of the section.

As a consequence we must have $\theta''(x_3)+\theta(x_3)=0$, from which $\theta(x_3)=b\cos x_3+c\sin x_3$ with b,c (arbitrary) constants. Then integrating (29), we have $\chi=-\cot x_2(b\sin x_3-c\cos x_3)+d$, where d is a new arbitrary constant.

The most general way of satisfying the fundamental equations in the present case is therefore given by the formula

$$\eta_1=a\ ,\ \eta_2=b\cos x_3+c\sin x_3\ ,\ \eta_3=\cot x_2(-b\sin x_3+c\cos x_3)+d\ ,$$

with a,b,c,d arbitrary constants.

Thus the complete group of motions of the present space is the 4-parameter group generated by the infinitesimal transformations

$$\begin{aligned} X_1f&=\frac{\partial f}{\partial x_3}\ ,\ X_2f=\sin x_3\frac{\partial f}{\partial x_2}+\cot x_2\cos x_3\frac{\partial f}{\partial x_3}\ ,\\ X_3f&=\cos x_3\frac{\partial f}{\partial x_2}-\cot x_2\sin x_3\frac{\partial f}{\partial x_3}\ ,\ X_4f=\frac{\partial f}{\partial x_1}\ ,\end{aligned}$$

whose composition is given therefore in the equations

$$\begin{aligned}&[X_1,X_2]f=X_3f\ ,\ [X_1,X_3]f=-X_2f\ ,\ [X_2,X_3]f=X_1f\ ,\\ &[X_1,X_4]f=[X_2,X_4]f=[X_3,X_4]f=0\ .\end{aligned}$$

The form of the line element $ds^2=dx_1^2+dx_2^2+\sin^2x_2\,dx_3^2$ already renders *a priori* evidence that, other than the ∞^3 motions which correspond to the sliding of each surface $x_1=constant$ into itself, there exists here a group G_1 with finite equations $x_1'=x_1+constant$, $x_2'=x_2$, $x_3'=x_3$.

But our calculations show that this G_4 is also the *complete* group of motions. Such a group G_4 is clearly transitive; furthermore it is *systatic* since the motions that leave a point of the space fixed also leave fixed all the points of that geodesic (x_1) which passes through it, so that these geodesics are the *systatic varieties* of the group. The whole space can be freely rotated around each one of these, but no other rotation is possible.

10 The group of motions of the space: $ds^2=dx_1^2+\varphi^2(x_1)\,(dx_2^2+e^{2x_2}dx_3^2)$.

The fundamental equations (A) are translated by the present space into the following:[20]

$$\frac{\partial\eta_1}{\partial x_1}=0\ , \tag{30}$$

$$\frac{\partial\eta_2}{\partial x_1}=-\frac{1}{\varphi^2}\frac{\partial\eta_1}{\partial x_2}\ , \tag{31a}$$

$$\frac{\partial\eta_2}{\partial x_2}=-\frac{\varphi'}{\varphi}\eta_1\ , \tag{31b}$$

$$\frac{\partial\eta_3}{\partial x_1}=-\frac{e^{-2x_2}}{\varphi^2}\frac{\partial\eta_1}{\partial x_3}\ , \tag{32a}$$

[20]In the original paper, eq. (31b) had $\partial\eta_2/\partial x_3$ on the l.h.s., and eq. (32a) had $\partial\eta_1/\partial x_2$ on the r.h.s., both of which were incorrect. Correction after the *Opere* [Editor].

$$\frac{\partial \eta_3}{\partial x_3} = -\frac{\varphi'}{\varphi}\eta_1 - \eta_2 \ , \tag{32b}$$
$$\frac{\partial \eta_2}{\partial x_3} + e^{2x_2}\frac{\partial \eta_3}{\partial x_2} = 0 \ . \tag{33}$$

Eliminating by differentiation η_2 from (31) and η_3 from (32) we find

$$\frac{\partial^2 \eta_1}{\partial x_2^2} = (\varphi''\varphi - \varphi'^2)\,\eta_1 \ ,$$
$$\frac{\partial^2 \eta_1}{\partial x_3^2} = e^{2x_2}(\varphi''\varphi - \varphi'^2)\,\eta_1 - e^{2x_2}\frac{\partial \eta_1}{\partial x_2} \ ,$$

from which (assuming $\eta_1 \neq 0$) one derives as usual $(\varphi''\varphi - \varphi'^2) = c$ (constant), so that

$$\frac{\partial^2 \eta_1}{\partial x_2^2} \;=\; c\eta_1 \ , \tag{34}$$
$$\frac{\partial^2 \eta_1}{\partial x_3^2} \;=\; e^{2x_2}\left(c\eta_1 - \frac{\partial \eta_1}{\partial x_2}\right) \ . \tag{35}$$

Integrating (31a) and (32a) with respect to x_1 we obtain:

$$\eta_2 = -\frac{\partial \eta_1}{\partial x_2}\int \frac{dx_1}{\varphi^2(x_1)} + \psi(x_2, x_3) \ , \tag{36}$$
$$\eta_3 = -e^{-2x_2}\frac{\partial \eta_1}{\partial x_3}\int \frac{dx_1}{\varphi^2(x_1)} + \chi(x_2, x_3) \ , \tag{37}$$

and substituting into (33) we have

$$2\left(\frac{\partial^2 \eta_1}{\partial x_2 \partial x_3} - \frac{\partial \eta_1}{\partial x_3}\right)\int \frac{dx_1}{\varphi^2(x_1)} = \frac{\partial \psi}{\partial x_3} + e^{2x_2}\frac{\partial \chi}{\partial x_2} \ .$$

Applying the usual observation, we deduce from this

$$\frac{\partial^2 \eta_1}{\partial x_2 \partial x_3} = \frac{\partial \eta_1}{\partial x_3} \ .$$

Differentiating this with respect to x_2 and comparing with (34) differentiated with respect to x_3, it follows that $(c-1)\,\partial\eta_1/\partial x_3 = 0$, from which it follows that $c = 1$ or $\partial\eta_1/\partial x_3 = 0$.

We treat the first case in this section. The equation $\varphi''\varphi - \varphi'^2 = 1$ differentiated gives $\varphi'' = k\varphi$, (k constant), so that $\varphi'^2 = k\varphi^2 - 1$. The constant k will necessarily be positive and, putting $k = 1/R^2$ and neglecting the additive constant in x_1, we will have $\varphi(x_1) = R\cosh(x_1/R)$. In such a case the space has the line element

$$ds^2 = dx_1^2 + R^2\cosh^2(x_1/R)\,(dx_2^2 + e^{2x_2}\,dx_3^2)$$

and is of constant negative curvature $K = -1/R^2$. Its complete group of motions is a G_6.

11 The group G_4 of motions of the space: $ds^2 = dx_1^2 + dx_2^2 + e^{2x_2}\, dx_3^2$.

We continue the discussion of the previous section assuming now $\partial\eta_1/\partial x_3 = 0$.

Equations (34), (35) give[21] $\partial^2\eta_1/\partial {x_2}^2 = c\eta_1$, $\partial\eta_1/\partial x_2 = c\eta_1$ from which $c^2 = c$ and consequently $c = 0$, the case $c = 1$ having already been discussed in §10. So we then have $\eta_1 = a$ (constant), and (36), (37) become $\eta_2 = \psi(x_2, x_3)$, $\eta_3 = \chi(x_2, x_3)$, while the equations at the beginning of §10 give

$$\frac{\partial\psi}{\partial x_2} + a\frac{\varphi'}{\varphi} = 0 \ , \ \frac{\partial\chi}{\partial x_3} + a\frac{\varphi'}{\varphi} + \psi = 0 \ , \ \frac{\partial\psi}{\partial x_3} + e^{2x_2}\frac{\partial\chi}{\partial x_2} = 0 \ .$$

We conclude from this that $\varphi' = k\varphi$ (k constant), from which it follows that

$$\begin{aligned} &\psi = -akx_2 + \theta(x_3) \ , \\ &\frac{\partial\chi}{\partial x_2} = -e^{2x_2}\theta'(x_3) \ , \ \frac{\partial\chi}{\partial x_3} = -ak + akx_2 - \theta(x_3) \ . \end{aligned}$$

Writing the integrability condition for these last two equations, we find $e^{-2x_2}\theta''(x_3) + ak = 0$, from which $k = 0$, $\theta''(x_3) = 0$ establishing the most general values of η_1, η_2, η_3 to be:

$$\eta_1 = a \ , \ \eta_2 = bx_3 + c \ , \ \eta_3 = \frac{b}{2}(e^{-2x_2} - x_3^2) - cx_3 + d \ ,$$

with a, b, c, d arbitrary constants. By replacing the space with a similar space, one can make $\varphi(x_1) = 1$ as in §9 and one therefore has the line element

$$ds^2 = dx_1^2 + dx_2^2 + e^{2x_2}\, dx_3^2 \ .$$

Therefore here also as in §9, the complete group of motions is a G_4. Its infinitesimal transformation generators are:

$$\begin{aligned} &X_1 f = \frac{\partial f}{\partial x_3} \ , \ X_2 f = -\frac{\partial f}{\partial x_2} + x_3\frac{\partial f}{\partial x_3} \ , \\ &X_3 f = x_3\frac{\partial f}{\partial x_2} + \frac{1}{2}(e^{-2x_2} - x_3^2)\frac{\partial f}{\partial x_3} \ , \ X_4 f = \frac{\partial f}{\partial x_1} \ , \end{aligned}$$

and have the composition

$$\begin{aligned} &[X_1, X_2]f = X_1 f \ , \ [X_2, X_3]f = X_3 f \ , \ [X_3, X_1]f = X_2 f \ , \\ &[X_1, X_4]f = [X_2, X_4]f = [X_3, X_4]f = 0 \ . \end{aligned}$$

The properties of the group are entirely similar to those already described for the group in §10. However, the two corresponding spaces belong to essentially different types, a fact established by the observation that the surfaces orthogonal to the systatic geodesics (x_1) are surfaces of constant positive curvature for the space of §10, while for the present space they are of constant negative curvature.

We summarize these last results obtained here in the theorem:

[21] Bianchi used the ordinary derivative d instead of ∂ in both equations here, correction by the Editor.

If a space of three dimensions admits an intransitive group G_3 of motions, its line element is reducible to one of the 3 standard forms:

$$ds^2 = dx_1^2 + \varphi^2(x_1)\,(dx_2^2 + dx_3^2) \ ,$$
$$ds^2 = dx_1^2 + \varphi^2(x_1)\,(dx_2^2 + \sin^2 x_2\, dx_3^2) \ ,$$
$$ds^2 = dx_1^2 + \varphi^2(x_1)\,(dx_2^2 + e^{2x_2}\, dx_3^2)$$

and in general the complete group of motions is exactly a 3-parameter group. The only exceptions are the two special spaces

$$ds^2 = dx_1^2 + dx_2^2 + \sin^2 x_2\, dx_3^2 \ ,$$
$$ds^2 = dx_1^2 + dx_2^2 + e^{2x_2}\, dx_3^2 \ ,$$

each with a 4-parameter group of motions, and the spaces of constant curvature with 6-parameter groups.

12 Spaces with a transitive group G_3 of motions.

Having exhausted the study of spaces which admit an intransitive G_3 of motions in the previous sections, let us now turn to the treatment of the spaces with a transitive group of motions.

In this section, we begin to establish in general that given any group G_3 whatsoever, transitive over 3 variables x_1, x_2, x_3, there always exist some spaces of 3 dimensions which admit it as a group of motions. In fact we establish more generally the analogous result for any number n of dimensions with the theorem:

Given any transitive group of n parameters over n variables:

$$G_n \equiv (X_1 f, X_2 f, \ldots, X_n f) \ ,$$

it is always possible to find spaces of n dimensions which admit it as a group of motions.[22]

To avoid confusion, however, we state immediately that the spaces S_n so determined may very well admit a larger group as the *complete* group of motions, as the case $n = 2$ has already shown (see §3).

We assume in general

$$X_\alpha f = \sum_i^{1\ldots n} \xi_i^{(\alpha)} \frac{\partial f}{\partial x_i} \ , \ (\alpha = 1, 2, \ldots, n) \ ,$$

and one will have:

$$[X_\alpha, X_\beta] f = \sum_\gamma c_{\alpha\beta\gamma}\, X_\gamma f \ , \tag{38}$$

where $c_{\alpha\beta\gamma}$ are the *constants of composition*. Furthermore, since the group is assumed to be transitive, the determinant

$$|\xi^{\alpha)}| = \begin{vmatrix} \xi_1^{(1)} & \xi_2^{(1)} & \cdots & \xi_n^{(1)} \\ \xi_1^{(2)} & \xi_2^{(2)} & \cdots & \xi_n^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ \xi_1^{(n)} & \xi_2^{(n)} & \cdots & \xi_n^{(n)} \end{vmatrix}$$

[22] If the group is not simply transitive the theorem does not hold in general as is already shown by the theorem at the beginning of §2.

will be different from zero.

Here the coefficients $\xi_i^{(\alpha)}$ are given as functions of x and we have to determine the coefficients a_{ik} of the differential form $ds^2 = \sum_{i,k} a_{ik}\, dx_i dx_k$ so that it admits the group G_n, in other words so that the fundamental equations (A) are satisfied by all the n transformations $X_\alpha f$. To determine the a_{ik} we therefore have the $n^2(n+1)/2$ partial differential equations

$$X_\alpha(a_{ik}) + \sum_r \left(a_{ir}\frac{\partial \xi_r^{(\alpha)}}{\partial x_k} + a_{kr}\frac{\partial \xi_r^{(\alpha)}}{\partial x_i} \right) = 0 \ , \qquad \text{(D)}$$
$$(\alpha, i, k, = 1, 2, 3, \ldots, n) \ .$$

If in (D) we fix i, k and let α take the n values $1, 2, \ldots, n$, we can solve the resulting equations for the n first derivatives of a_{ik} since by hypothesis $|\xi_i^{(\alpha)}| \neq 0$. We therefore have a system of linear and homogeneous total differential equations for our unknowns a_{ik}. We show that this system is completely integrable, for which it suffices to prove that by writing two of the equations (D) for the same unknown a_{ik}:

$$X_\alpha(a_{ik}) + \sum_r a_{ir}\frac{\partial \xi_r^{(\alpha)}}{\partial x_k} + \sum_r a_{kr}\frac{\partial \xi_r^{(\alpha)}}{\partial x_i} = 0 \ ,$$
$$X_\beta(a_{ik}) + \sum_s a_{is}\frac{\partial \xi_s^{(\beta)}}{\partial x_k} + \sum_s a_{ks}\frac{\partial \xi_s^{(\beta)}}{\partial x_i} = 0 \ ,$$

and if the operation X_α is performed on the second of these, the operation X_β on the first of these, and one subtracts the results making use of the same equation (D), the result is an identity. Using (38) on this relation one obtains in this way first[23]

$$\begin{aligned} &\sum_\gamma c_{\alpha\beta\gamma} X_\gamma(a_{ik}) + \sum_s X_\alpha(a_{is})\frac{\partial \xi_s^{(\beta)}}{\partial x_k} + \sum_s X_\alpha(a_{ks})\frac{\partial \xi_s^{(\beta)}}{\partial x_i} \\ &\quad - \sum_r X_\beta(a_{ir})\frac{\partial \xi_r^{(\alpha)}}{\partial x_k} - \sum_r X_\beta(a_{kr})\frac{\partial \xi_r^{(\alpha)}}{\partial x_i} \\ &\quad + \sum_r a_{ir}\left[X_\alpha\left(\frac{\partial \xi_r^{(\beta)}}{\partial x_k}\right) - X_\beta\left(\frac{\partial \xi_r^{(\alpha)}}{\partial x_k}\right)\right] \\ &\quad + \sum_r a_{kr}\left[X_\alpha\left(\frac{\partial \xi_r^{(\beta)}}{\partial x_i}\right) - X_\beta\left(\frac{\partial \xi_r^{(\alpha)}}{\partial x_i}\right)\right] = 0 \ . \end{aligned} \qquad (39)$$

Now from (38) itself one has

$$X_\alpha(\xi_r^{(\beta)}) - X_\beta(\xi_r^{(\alpha)}) = \sum_r c_{\alpha\beta\gamma}\, \xi_r^{(\gamma)} \ ,$$

which by differentiating with respect to x_k becomes

$$\begin{aligned} &X_\alpha\left(\frac{\partial \xi_r^{(\beta)}}{\partial x_k}\right) - X_\beta\left(\frac{\partial \xi_r^{(\alpha)}}{\partial x_k}\right) \\ &\qquad = \sum_\gamma c_{\alpha\beta\gamma}\frac{\partial \xi_r^{(\gamma)}}{\partial x_k} + \sum_s \left(\frac{\partial \xi_r^{(\alpha)}}{\partial x_s}\frac{\partial \xi_s^{(\beta)}}{\partial x_k} - \frac{\partial \xi_s^{(\alpha)}}{\partial x_k}\frac{\partial \xi_r^{(\beta)}}{\partial x_s} \right) \ , \end{aligned}$$

[23] In the last line of eq. (39), $\sum_i$ was corrected to $\sum_r$ [Editor].

and similarly

$$X_\alpha \left(\frac{\partial \xi_r^{(\beta)}}{\partial x_i} \right) - X_\beta \left(\frac{\partial \xi_r^{(\alpha)}}{\partial x_i} \right)$$
$$= \sum_\gamma c_{\alpha\beta\gamma} \frac{\partial \xi_r^{(\gamma)}}{\partial x_i} + \sum_s \left(\frac{\partial \xi_r^{(\alpha)}}{\partial x_s} \frac{\partial \xi_s^{(\beta)}}{\partial x_i} - \frac{\partial \xi_r^{(\beta)}}{\partial x_s} \frac{\partial \xi_s^{(\alpha)}}{\partial x_i} \right) \ .$$

If in the first 5 terms of (39) we introduce the values of $X(a)$ given by (D) and in the last 2 terms the values calculated above, we see that it is converted into an identity. We conclude from this that the system of total differential equations for the a_{ik} is completely integrable and we can therefore give the initial values of the a_{ik} arbitrarily at a point of the space S_n. So if we choose them in such a way that the conditions (of inequality) making the differential form positive definite are *initially* satisfied, they will remain so in a certain neighborhood of that point and we will therefore have defined a space of n dimensions which admits the group G_n as a group of motions.

13 Preliminary classification of the various types of G_3.

With the general considerations of the previous sections we are assured that to any G_3 transitive over 3 variables always correspond spaces of 3 dimensions which admit it as a group of motions. It is not true, however, and is not even true in all cases, that the *complete* group of motions of the space obtained is indeed the given G_3. It will be seen instead that there are certain compositions of the G_3 which necessarily imply the existence of a larger group of motions.[24] Furthermore we wish to establish for any possible type of G_3 a corresponding canonical form for the line element, by performing the integration which we have only described in the previous section. As the basis of our calculations we take the classification given by Lie of the possible compositions of groups of 3 parameters.[25] But here an essential warning is necessary for us. In the classification of Lie there is no way for us to distinguish between real and complex, whereas in this study we wish to report only on real groups and their real subgroups: we will therefore have to subdivide into more types some types which are a single type from the general point of view of Lie.

Without repeating the discussion given by Lie (*ibid.*), it will suffice to point out that, considering first the *integrable* groups, to the 6 types classified by Lie according to the following compositions

$$\begin{aligned}
&\text{(Type I)} && [X_1, X_2]f = [X_1, X_3]f = [X_2, X_3]f = 0 \ , \\
&\text{(Type II)} && [X_1, X_2]f = [X_1, X_3]f = 0 \ , \ [X_2, X_3]f = X_1 f \ , \\
&\text{(Type III)} && [X_1, X_2]f = 0 \ , \ [X_1, X_3]f = X_1 f \ , \\
& && \quad [X_2, X_3]f = 0 \ , \\
&\text{(Type IV)} && [X_1, X_2]f = 0 \ , \ [X_1, X_3]f = X_1 f \ , \\
& && \quad [X_2, X_3]f = X_1 f + X_2 f \ ,
\end{aligned}$$

[24] This happens for the groups G_3 of types I, II, III, V in the classification of the present section.

[25] S. Lie-F. Engel, Vol. III, p. 713 and S. Lie-G. Scheffers, *Vorlesungen über continuierliche Gruppen* (1893), p. 565.

$$
\begin{aligned}
&\text{(Type V)} \qquad [X_1, X_2]f = 0 \ , \ [X_1, X_3]f = X_1 f \ , \\
&\qquad\qquad\qquad [X_2, X_3]f = X_2 f \ , \\
&\text{(Type VI)} \qquad [X_1, X_2]f = 0 \ , \ [X_1, X_3]f = X_1 f \ , \\
&\qquad\qquad\qquad [X_2, X_3]f = hX_2 f \ , \ (h \neq 0, 1) \ ,
\end{aligned}
$$

we must add a seventh type with the composition

$$
\begin{aligned}
&\text{(Type VII)} \qquad [X_1, X_2]f = 0 \ , \ [X_1, X_3]f = X_2 f \ , \\
&\qquad\qquad\qquad [X_2, X_3]f = -X_1 f + hX_2 f \ ,
\end{aligned}
$$

where the constant h satisfies the inequality $0 \leq h < 2$.[26]

From our real point of view this composition in effect differs from all of the previous ones in that, while in the first 6 types one has at least a real *invariant* subgroup G_1, in type VII, however, no such *real* subgroup exists.[27]

Furthermore, it is necessary to observe that in the new composition VII the constant h is truly *essential*, namely that if there is a second group $(Y_1 f, Y_2 f, Y_3 f)$ of composition

$$
\begin{aligned}
&[Y_1, Y_2]f = 0 \ , \ [Y_1, Y_3]f = Y_2 f \ , [Y_2, Y_3]f = -Y_1 f + kY_2 f \ , \\
&\qquad (0 \leq k < 2) \ ,
\end{aligned}
\tag{39}
$$

if $k \neq h$, then the two groups cannot be put into an isomorphic correspondence. Indeed if this occurred and we indicate by $\bar{X}_1 f$, $\bar{X}_2 f$, $\bar{X}_3 f$, the infinitesimal transformations of the first group which correspond respectively to $Y_1 f$, $Y_2 f$, $Y_3 f$ in the second, then $\bar{X}_1 f$, $\bar{X}_2 f$ must be constructed only with $X_1 f$, $X_2 f$ since both pairs of transformations belong to the derived group. We assume therefore:

$$
\begin{aligned}
&\bar{X}_1 f = aX_1 f + \beta X_2 f \ , \ \bar{X}_2 f = \gamma X_1 f + \delta X_2 f \ , \\
&\qquad \bar{X}_3 f = aX_1 f + bX_2 f + cX_3 f \ ,
\end{aligned}
$$

and from the assumed relations of composition we find the following relations among the constants $\alpha, \beta, \gamma, \delta, c$:

$$
\begin{aligned}
&\gamma + \beta c = 0 \ , \ \delta - \alpha c - h\beta c = 0 \ , \\
&\alpha - k\gamma - c\delta = 0 \ , \ \beta - k\delta + c\gamma + hc\delta = 0 \ ,
\end{aligned}
$$

so that

$$
\alpha = c(\delta - k\beta) \ , \ \gamma = -\beta c \ , \ \begin{cases} \beta(1 - c^2) + (hc - k)\delta = 0 \ , \\ \beta c(h - kc) + (c^2 - 1)\delta = 0 \ . \end{cases}
$$

[26] The sign of h is not essential, as one sees by simultaneously changing the signs of $X_2 f$, $X_3 f$.

[27] If $Yf = \alpha_1 X_1 f + \alpha_2 X_2 f + \alpha_3 X_3 f$ were the infinitesimal transformation generator of such a subgroup, the three infinitesimal transformations $[Y, X_1]f$, $[Y, X_2]f$, $[Y, X_3]f$, would have to differ from Yf only by a constant factor. It follows immediately from this that $\alpha_3 = 0$, and then from

$$
[Y, X_3]f = \alpha_1 X_2 f + \alpha_2(-X_1 f + hX_2 f) = \rho(\alpha_1 X_1 f + \alpha_2 X_2 f) \ ,
$$

we obtain $\rho\alpha_1 + \alpha_2 = 0$, $\rho\alpha_2 - \alpha_1 - h\alpha_2 = 0$, so that $\rho^2 - h\rho + 1 = 0$, an equation with complex roots since $h^2 < 4$.

From these last two equations, since both β and δ cannot be simultaneously zero, it follows that c satisfies the 4th degree equation

$$c^4 - hkc^3 + (h^2 + k^2 - 2)c^2 - hkc + 1 = 0 \ ;$$

but then the determinant $\alpha\delta - \beta\gamma$ (since $c^2 \neq 1$ because $k \neq \pm h$) would have to be zero, but that is absurd.

There remains finally to consider the case in which the group G_3 is not integrable. For these groups Lie assigned the single type

$$\text{(Type VIII)} \qquad \begin{aligned} &[X_1, X_2]f = X_1 f \ , \ [X_1, X_3]f = 2X_2 f \ , \\ &[X_2, X_3]f = X_3 f \ , \end{aligned}$$

but we must add another:

$$\text{(Type IX)} \qquad \begin{aligned} &[X_1, X_2]f = X_3 f \ , \ [X_2, X_3]f = X_1 f \ , \\ &[X_3, X_1]f = X_2 f \ , \end{aligned}$$

which differs from the previous one only in that there does not exist a *real* 2-parameter subgroup in this last case.[28]

14 The groups of type I.

In the first seven types the group G_3 contains the Abelian 2-parameter subgroup of motions $G_2 \equiv (X_1 f, X_2 f)$. The considerations of §4 show that with respect to this G_2 the minimum invariant varieties are geodesically parallel surfaces of zero curvature. By assuming these as the coordinate surfaces $x_1 = constant$, we can furthermore make $X_1 f = \partial f/\partial x_2$, $X_2 f = \partial f/\partial x_3$ and the line element of the space will take the form

$$ds^2 = dx_1^2 + \alpha \, dx_2^2 + 2\beta \, dx_2 dx_3 + \gamma \, dx_3^2 \ , \tag{40}$$

with α, β, γ functions only of x_1. To determine the most general infinitesimal motion of this space the fundamental equations (A) give us the system

$$\begin{aligned} &\frac{\partial \eta_1}{\partial x_1} = 0 \ , \\ &\frac{\partial \eta_1}{\partial x_2} + \alpha \frac{\partial \eta_2}{\partial x_1} + \beta \frac{\partial \eta_3}{\partial x_1} = 0 \ , \\ &\frac{\partial \eta_1}{\partial x_3} + \beta \frac{\partial \eta_2}{\partial x_1} + \gamma \frac{\partial \eta_3}{\partial x_1} = 0 \ , \\ &\frac{1}{2}\alpha' \eta_1 + \alpha \frac{\partial \eta_2}{\partial x_2} + \beta \frac{\partial \eta_3}{\partial x_2} = 0 \ , \\ &\frac{1}{2}\gamma' \eta_1 + \beta \frac{\partial \eta_2}{\partial x_3} + \gamma \frac{\partial \eta_3}{\partial x_3} = 0 \ , \\ &\beta' \eta_1 + \alpha \frac{\partial \eta_2}{\partial x_3} + \beta \left(\frac{\partial \eta_2}{\partial x_2} + \frac{\partial \eta_3}{\partial x_3} \right) + \gamma \frac{\partial \eta_3}{\partial x_2} = 0 \ . \end{aligned} \tag{E}$$

[28] In the geometrical representation given by Lie on p. 718 of Vol. III, according to reciprocity in the plane with respect to a conic, one case is distinguished from the other by type VIII having a real conic and type IX a complex conic.

Now if we assume that there exists a third infinitesimal transformation $X_3 f = \xi_1 \partial f/\partial x_1 + \xi_2 \partial f/\partial x_2 + \xi_3 \partial f/\partial x_3$, which with $X_1 f, X_2 f$ generates a group G_3, we have in general

$$[X_1, X_3]f = aX_1 f + bX_2 f + cX_3 f \ ,$$
$$[X_2, X_3]f = a'X_1 f + b'X_2 f + c'X_3 f \ ,$$

with a, b, c, a', b', c' constants, and therefore the following 6 equations hold

$$\frac{\partial \xi_1}{\partial x_2} = c\xi_1 \ , \ \frac{\partial \xi_2}{\partial x_2} = c\xi_2 + a \ , \ \frac{\partial \xi_3}{\partial x_2} = c\xi_3 + b \ , \tag{41a}$$
$$\frac{\partial \xi_1}{\partial x_3} = c'\xi_1 \ , \ \frac{\partial \xi_2}{\partial x_3} = c'\xi_2 + a' \ , \ \frac{\partial \xi_3}{\partial x_3} = c'\xi_3 + b' \ , \tag{41b}$$

and since we furthermore assume that the group $(X_1 f, X_2 f, X_3 f)$ is transitive, we will have $\xi_1 \neq 0$.

Now the system (E) has to be satisfied when the η are replaced by the ξ and so we will therefore have[29]

$$\begin{aligned}
&\frac{\partial \xi_1}{\partial x_1} = 0 \ , \\
&c\xi_1 + \alpha \frac{\partial \xi_2}{\partial x_1} + \beta \frac{\partial \xi_3}{\partial x_1} = 0 \ , \\
&c'\xi_1 + \beta \frac{\partial \xi_2}{\partial x_1} + \gamma \frac{\partial \xi_3}{\partial x_1} = 0 \ , \\
&\frac{1}{2}\alpha'\xi_1 + \alpha(c\xi_2 + a) + \beta(c\xi_3 + b) = 0 \ , \\
&\frac{1}{2}\gamma'\xi_1 + \beta(c'\xi_2 + d') + \gamma(c'\xi_3 + b') = 0 \ , \\
&\beta'\xi_1 + \alpha(c'\xi_2 + a') + \beta(c\xi_2 + c'\xi_3 + a + b') + \gamma(c\xi_3 + b) = 0 \ ,
\end{aligned} \tag{F}$$

These are the equations which will serve to solve for us the problem posed for the groups of the first seven types.

Meanwhile for type I, since the constants a, b, c, a', b', c' are all zero, the last three equations of (F), remembering that $\xi_1 \neq 0$, show that α, β, γ are constants and so the space is of zero curvature. Since then there do not exist spaces with an Abelian intransitive G_3 of motions, as results from the discussion of the previous sections and also if one wishes, from the same system (F) and from (41), we can state the result: *If a space of 3 dimensions admits a 3-parameter Abelian group of motions, it is of zero curvature and the group is the translation group.*

15 Digressions relative to spaces of n dimensions.

It will not be useless to observe that the preceding theorem holds for spaces of any number of dimensions, namely:

A space of n dimensions which admits an n-parameter Abelian group of translations is necessarily of zero curvature and the group is the translation group.

[29] In the first line of the equation, the original paper has $\partial\xi_1/\partial x_2$; correction based on the *Opere* [Editor].

To show this it is sufficient to appeal to the result established by Lie[30] namely the theorem that if r infinitesimal transformations $X_1f, X_2f, \ldots, X_rf$ over n variables $x_1, x_2, \ldots, x_n$ commute, i.e., one has $[X_i, X_j]f = 0$, $(i, k = 1, 2, \ldots, r)$ and among the r Xf does not exist any linear identity of the form

$$\sum_{i=1}^{r} \alpha_i(x_1, x_2, \ldots, x_n) X_i f = 0 \ ,$$

where the α are functions of the x, with a convenient transformation of variables they can be reduced to the form:

$$X_1 f = \frac{\partial f}{\partial x_1} \ , \ X_2 f = \frac{\partial f}{\partial x_2} \ , \ \ldots \ , \ X_r f = \frac{\partial f}{\partial x_r} \ ,$$

Therefore with $G_n \equiv (X_1f, X_2f, \ldots, X_nf)$ the hypothetical group, it will be enough to show that there does not exist among the $X_1f, X_2f, \ldots, X_nf$ an identity of the above form, namely that G_n is transitive, since then having reduced the group of motions to the canonical form $(\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n)$ by the fundamental equations (A) the coefficients a_{ik} of the line element will be independent of all the x, namely absolute constants, and so we will have a space of zero curvature. Now we assume that among the first s of the X_if: $X_1f, X_2f, \ldots, X_sf$ does not exist any linear identity of the form mentioned above (and we will have by the theorem of §2: $s \geq 2$), while one has $X_{s+1}f = \xi_1 X_1 f + \xi_2 X_2 f + \cdots + \xi_s X_s f$, the ξ being functions of the x which are *not all* constants. By the cited theorem of Lie we can assume

$$X_1 f = \frac{\partial f}{\partial x_1} \ , \ X_2 f = \frac{\partial f}{\partial x_2} \ , \ \ldots \ , \ X_s f = \frac{\partial f}{\partial x_s} \ ,$$

and we will have

$$X_{s+1} f = \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \cdots + \xi_s \frac{\partial f}{\partial x_s} \ .$$

First the conditions

$$[X_{s+1}, X_1]f = 0 \ , \ [X_{s+1}, X_2]f = 0 \ , \ \ldots \ , [X_{s+1}, X_s]f = 0$$

show that the ξ do not depend on the first s variables $x_1, x_2, \ldots, x_s$. Secondly, the fundamental equations (A), where one fixes k and sets $i = 1, 2, \ldots, s$, give

$$\sum_{r=1}^{s} a_{ir} \frac{\partial \xi_r}{\partial x_k} = 0 \ , (i = 1, 2, 3, \ldots, s) \ .$$

Now the determinant $\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1s} \\ \cdot & \cdot & \cdot & \cdot \\ a_{s1} & a_{s2} & \ldots & a_{ss} \end{vmatrix}$ is different from zero, and also positive since the differential form $\sum_{i,k} a_{ik}\, dx_i dx_k$ is positive-definite, so that we have the result that $\xi_1, \xi_2, \ldots, \xi_s$ are absolute constants, which is absurd.

[30]See S. Lie-F. Engel, Vol. I, p. 339.

16 The groups of type II:
$[X_1, X_2]f = [X_1, X_3]f = 0,\ [X_2, X_3]f = X_1 f.$

Applying the general method described in §14, we must now set $a = b = c = 0$, $a' = 1$, $b' = c' = 0$.

From (41) and the first of (F) one then sees that ξ_1 must be a constant, so we set $\xi_1 = -1/h$, and the last three equations of (F) give us $\alpha' = 0$, $\beta' = h\alpha$, $\gamma' = 2h\beta$, from which by integrating

$$\alpha = k^2 \ , \ \beta = hk^2 x_1 + l \ , \ \gamma = h^2k^2x_1^2 + 2hlx_1 + m \ ,$$

with k, l, m new constants.[31] The line element of the space therefore has the form[32]

$$ds^2 = dx_1^2 + k^2\, dx_2^2 + 2(hk^2x_1 + l)\, dx_2dx_3 + (h^2k^2x_1^2 + 2hlx_1 + m)\, dx_3^2 \ .$$

Replacing x_2, x_3 respectively by x_2/k, x_3/k, we can write

$$ds^2 = dx_1^2 + dx_2^2 + 2(hx_1 + l/k^2)\, dx_2dx_3 + [(hx_1 + l/k^2)^2 + n^2]\, dx_3^2 \ , \tag{42}$$

having set $n^2 = m/k^2 - l^2/k^4$, a constant necessarily positive since $\alpha\gamma - \beta^2 > 0$.

If we put $hx_1 + l/k^2 = ny_1$, $x_2 = n/h\, y_2$, $x_3 = 1/h\, y_3$, (42) becomes

$$ds^2 = n^2/h^2\, [dy_1^2 + dy_2^2 + 2y_1\, dy_2dy_3 + (y_1^2 + 1)\, dx_3^2] \ .$$

By substituting a similar space, we can therefore assume as the standard form for the line element:

$$ds^2 = dx_1^2 + dx_2^2 + 2x_1\, dx_2dx_3 + (x_1^2 + 1)\, dx_3^2 \ . \tag{43}$$

This space certainly admits a transitive group G_3 of motions of type II, but as we now show, its complete group of motions is a G_4 of which the original G_3 is not the derived subgroup.

To determine the most general infinitesimal motion $Xf = \eta_1\partial f/\partial x_1 + \eta_2\partial f/\partial x_2 + \eta_3\partial f/\partial x_3$ of the space (43) it suffices to apply the equations (E) of §14, which here become:

$$\frac{\partial\eta_1}{\partial x_1} = 0 \ , \tag{44}$$

$$\frac{\partial\eta_1}{\partial x_2} + \frac{\partial\eta_2}{\partial x_1} + x_1\frac{\partial\eta_3}{\partial x_1} = 0 \ , \tag{45a}$$

$$\frac{\partial\eta_1}{\partial x_3} + x_1\frac{\partial\eta_2}{\partial x_1} + (x_1^2 + 1)\frac{\partial\eta_3}{\partial x_1} = 0 \ , \tag{45b}$$

$$\frac{\partial\eta_2}{\partial x_3} + x_1\frac{\partial\eta_3}{\partial x_2} = 0 \ , \tag{46}$$

$$x_1\eta_1 + x_1\frac{\partial\eta_2}{\partial x_3} + (x_1^2 + 1)\frac{\partial\eta_3}{\partial x_3} = 0 \ , \tag{47}$$

$$\eta_1 + \frac{\partial\eta_2}{\partial x_3} + x_1\left(\frac{\partial\eta_2}{\partial x_2} + \frac{\partial\eta_3}{\partial x_3}\right) + (x_1^2 + 1)\frac{\partial\eta_3}{\partial x_2} = 0 \ . \tag{48}$$

[31] We have indicated the value of α by k^2 since it must be positive.

[32] The $h^2k^2x_1^2$ in the coefficient of dx_3^2 is a correction based on the *Opere*, the original had $h^2k^2x_1$ here [Editor].

Solving (45) for $\partial\eta_2/\partial x_1$ and $\partial\eta_3/\partial x_1$ and integrating with respect to x_1 with the observation that by (44) η_1 does not depend on x_1, we have

$$\eta_2 = \frac{x_1^2}{2}\frac{\partial\eta_1}{\partial x_3} - \left(\frac{x_1^2}{3} + x_1\right)\frac{\partial\eta_1}{\partial x_2} + \psi(x_2, x_3) \ ,$$
$$\eta_3 = \frac{x_1^2}{2}\frac{\partial\eta_1}{\partial x_2} - x_1\frac{\partial\eta_1}{\partial x_3} + \chi(x_2, x_3) \ .$$

By substituting these values of η_2, η_3 into (46) we obtain a 3rd degree polynomial in x_1 which must be identically zero; from this we then deduce:

$$\frac{\partial^2\eta_1}{\partial {x_2}^2} = \frac{\partial^2\eta_1}{\partial x_2\partial x_3} = 0 \ , \ \frac{\partial\psi}{\partial x_2} = \frac{\partial\chi}{\partial x_2} = 0 \ .$$

Proceeding similarly with (47) we finally find

$$\frac{\partial^2\eta_1}{\partial {x_3}^2} = 0 \ , \ \frac{\partial\psi}{\partial x_3} = -\eta_1 \ , \ \frac{\partial\chi}{\partial x_3} = 0 \ ,$$

so that

$$\frac{\partial\eta_1}{\partial x_2} = -\frac{\partial^2\psi}{\partial x_2\partial x_3} = 0 \ .$$

Therefore η_1 will be a linear function depending only on x_3, so we set $\eta_1 = ax_3 + b$, and we have $\psi = -\frac{1}{2}ax_3^2 - bx_3 + c$, $\chi = d$, with a, b, c, d arbitrary constants. With the corresponding values of η_1, η_2, η_3:

$$\eta_1 = ax_3 + b \ , \ \eta_2 = \frac{1}{2}ax_1^2 - \frac{1}{2}ax_3^2 - bx_3 + c \ , \ \eta_3 = -ax_1 + d \ ,$$

(48) is also satisfied no matter what values a, b, c, d take. So the complete group of motions of the space (43) is the G_4 generated by the four infinitesimal transformations

$$X_1f = \frac{\partial f}{\partial x_2} \ , \ X_2f = \frac{\partial f}{\partial x_3} \ , \ X_3f = -\frac{\partial f}{\partial x_1} + x_3\frac{\partial f}{\partial x_2} \ ,$$
$$X_4f = x_3\frac{\partial f}{\partial x_1} + \frac{1}{2}(x_1^2 - x_3^2)\frac{\partial f}{\partial x_2} - x_1\frac{\partial f}{\partial x_3} \ ,$$

whose composition is expressed by the equations

$$[X_1, X_2]f = 0 \ , \ [X_1, X_3]f = 0 \ , \ [X_1, X_4]f = 0 \ ,$$
$$[X_2, X_3]f = X_1f \ , \ [X_2, X_4]f = -X_3f \ , \ [X_3, X_4]f = X_2f \ .$$

As one can see, its derived group is the transitive group $G_3 \equiv (X_1f, X_2f, X_3f)$ of type II. The three transformations X_1f, X_2f, X_3f are not related by any linear identity while one has

$$X_4f = \frac{1}{2}(x_2^2 + x_3^2)X_1f - x_1X_2f - x_3X_3f \ ,$$

and since the coefficients of this relation are functions only of x_1, x_3, we conclude from this[33] that the group is systatic and the systatic varieties are the coordinate lines (x_2).

[33] S. Lie-F. Engel, Vol. I, p. 502.

It is clear geometrically that these systatic lines are geodesics of the space,[34] and this statement also follows immediately from the form (43) of the line element of the space. The properties of the group are similar to those described in §9, §11 for the groups of the spaces:

$$ds^2 = dx_1^2 + dx_2^2 + \sin^2 x_2 \, dx_3^2 \, ,$$
$$ds^2 = dx_1^2 + dx_2^2 + e^{2x_2} \, dx_3^2 \, .$$

However, the different nature of these spaces follows immediately upon examining the compositions of their groups of motions. While for these latter spaces the derived group is an intransitive and simple G_3, for the space (43) the derived group is a transitive and integrable G_3. We also observe an essential difference geometrically since for those spaces discussed previously the systatic geodesics admit a family of orthogonal surfaces, which does not occur for the space (43).[35]

Finally we observe that it is easy to write the equations of the present group G_4 in finite terms. Those of the derived subgroup are given by the equations:

$$x_1' = x_1 + a_1 \; , \; x_2' = x_2 - a_1 x_3 + a_2 - a_1 a_3 \; , \; x_3' = x_3 + a_3 \; ,$$

with parameters a_1, a_2, a_3. It now suffices to associate with these ∞^3 motions the group G_1 generated by the infinitesimal transformation whose finite equations are

$$x_1' = x_1 \cos t + x_3 \sin t \; , \; x_3' = -x_1 \sin t + x_3 \cos t \; ,$$
$$x_2' = \frac{1}{4}(x_1^2 - x_3^2) \sin(2t) - \frac{1}{2} x_1 x_3 \cos(2t) + x_2 - \frac{1}{2} x_1 x_3$$

and which represents a rotation around the geodesic $x_1 = 0$, $x_3 = 0$ by an angle easily seen to be t.

17 The groups of type III: $[X_1, X_2]f = 0 \; , \; [X_1, X_3]f = X_1 f, \; [X_2, X_3]f = 0$.

For the above composition we must set $a = 1$, $b = c = 0$, $a' = b' = c' = 0$ in the equations of §14, from which it again follows that ξ_1 is constant, so we set $\xi_1 = -1/h$, and the last 3 equations of (F) give us $\alpha' = 2h\alpha$, $\gamma' = 0$, $\beta' = h\beta$.

Integrating and choosing conveniently the variables x_2, x_3 we can make $\alpha = e^{2hx_1}$, $\beta = ne^{hx_1}$, $\gamma = 1$, with n a new constant, and by replacing the space by a similar one, we can set $h = 1$ and have as the standard form of the line element of the present space:

$$ds^2 = dx_1^2 + e^{2x_1} \, dx_2^2 + 2ne^{x_1} \, dx_2 dx_3 + dx_3^2 \, . \tag{49}$$

One will observe that if $n = 0$ one again obtains the space of §11. Since $\alpha\gamma - \beta^2$ has to be positive, we will have $n^2 < 1$, and since the sign of n is not essential (as one sees by changing x_2 into $-x_2$, for example), we can assume $0 < n < 1$.

[34] In fact take two arbitrary points P, Q on a coordinate line (x_2). Those transformations of the space which leave P fixed also leave Q fixed and consequently all the points of the geodesic which joins P to Q, which therefore must coincide with the coordinate line (x_2).

[35] To determine the possible surfaces orthogonal to the geodesic (x_2) one would have the total differential equation $dx_2 + x_1 dx_3 = 0$ which is not integrable.

We will see that also in the case $n > 0$ as for $n = 0$, the space (49) has a 4-parameter group of motions.

The equations (E) §14 here become

$$\frac{\partial \eta_1}{\partial x_1} = 0 \ , \tag{50}$$

$$\frac{\partial \eta_1}{\partial x_2} + e^{2x_1}\frac{\partial \eta_2}{\partial x_1} + ne^{x_1}\frac{\partial \eta_3}{\partial x_1} = 0 \ , \tag{51a}$$

$$\frac{\partial \eta_1}{\partial x_3} + ne^{x_1}\frac{\partial \eta_2}{\partial x_1} + \frac{\partial \eta_3}{\partial x_1} = 0 \ , \tag{51b}$$

$$e^{x_1}\eta_1 + e^{x_1}\frac{\partial \eta_2}{\partial x_2} + n\frac{\partial \eta_3}{\partial x_2} = 0 \ , \tag{52}$$

$$ne^{x_1}\frac{\partial \eta_2}{\partial x_3} + \frac{\partial \eta_3}{\partial x_3} = 0 \ , \tag{53}$$

$$ne^{x_1}\eta_1 + e^{2x_1}\frac{\partial \eta_2}{\partial x_3} + ne^{x_1}\left(\frac{\partial \eta_2}{\partial x_2} + \frac{\partial \eta_3}{\partial x_3}\right) + \frac{\partial \eta_3}{\partial x_2} = 0 \ . \tag{54}$$

Solving (51) and integrating with respect to x_1 we obtain:

$$\eta_2 = \frac{-ne^{-x_1}}{1-n^2}\frac{\partial \eta_1}{\partial x_3} + \frac{e^{-2x_1}}{2(1-n^2)}\frac{\partial \eta_1}{\partial x_2} + \psi(x_2, x_3) \ ,$$

$$\eta_3 = \frac{-ne^{-x_1}}{1-n^2}\frac{\partial \eta_1}{\partial x_2} - \frac{x_1}{1-n^2}\frac{\partial \eta_1}{\partial x_3} + \chi(x_2, x_3) \ .$$

Substituting into (52), (53), (54) we conclude that

$$\frac{\partial \eta_1}{\partial x_3} = 0 \ , \ \frac{\partial^2 \eta_1}{\partial {x_2}^2} = 0 \ , \ \frac{\partial \psi}{\partial x_3} = \frac{\partial \chi}{\partial x_2} = \frac{\partial \chi}{\partial x_3} = 0 \ , \ \frac{\partial \psi}{\partial x_2} = -\eta_1 \ ,$$

from which

$$\eta_1 = ax_2 + b \ , \ \eta_2 = \frac{ae^{-2x_1}}{2(1-n^2)} - \frac{1}{2}ax_2^2 - bx_2 + c \ , \ \eta_3 = \frac{-ane^{-x_1}}{1-n^2} + d \ ,$$

with a, b, c, d arbitrary constants. The group of motions of the space (49) is therefore the G_4 generated by the 4 infinitesimal transformations:

$$X_1 f = \frac{\partial f}{\partial x_2} \ , \ X_2 f = \frac{\partial f}{\partial x_3} \ , \ X_3 f = \frac{\partial f}{\partial x_1} - x_2\frac{\partial f}{\partial x_2} \ ,$$

$$X_4 f = x_2\frac{\partial f}{\partial x_1} + \frac{1}{2}\left(\frac{e^{-2x_1}}{1-n^2} - x_2^2\right)\frac{\partial f}{\partial x_2} - \frac{ne^{-x_1}}{1-n^2}\frac{\partial f}{\partial x_3}$$

with the composition:

$$[X_1, X_2]f = 0 \ , \ [X_1, X_3]f = -X_1 f \ , \ [X_1, X_4]f = X_3 f \ ,$$

$$[X_2, X_3]f = 0 \ , \ [X_2, X_4]f = 0 \ , \ [X_3, X_4]f = -X_4 f \ .$$

The relation

$$X_4 f = \frac{1}{2}\left(\frac{e^{-2x_1}}{1-n^2} + x_2^2\right)X_1 f - \frac{ne^{-x_1}}{1-n^2}X_2 f + x_2 X_3 f$$

shows that the group is systatic and that the systatic varieties are the geodesics (x_3). These geodesics do not admit orthogonal surfaces except in the case $n = 0$ already considered in §11. We observe that the derived group is here the group $G_3 \equiv (X_1f, X_3f, X_4f)$, which is simply transitive and belongs to type VIII. In the space (49) we therefore also have an example of spaces corresponding to this type. To this purpose and for a better comparison with the results that we will establish in §28, we note the following transformation of the line element (49). Set:

$$x_1 = y_1 \ , \ x_2 = e^{-y_1}(y_2 - ny_3) \ , \ x_3 = y_3$$

and one will obtain

$$ds^2 = [1 + (y_2 - ny_3)^2]\, dy_1^2 + dy_2^2 + (1 - n^2)\, dy_3^2 - 2(y_2 - ny_3)\, dy_1 dy_2 \ . \qquad (49*)$$

18 Similarities of the groups of motions of two spaces of the type (49).

The line element (49) of the space of the previous section contains a constant n and we propose to demonstrate that this constant is truly essential, namely that to two distinct values of n $(0 < n < 1)$ correspond two spaces which are neither isometric nor similar.[36]

Assuming therefore a second line element of the form

$$ds^2 = dy_1{}^2 + e^{2y_1}\, dy_2{}^2 + 2me^{y_1}\, dy_2 dy_3 + dy_3{}^2 \ , \qquad (55)$$

where $m \neq n$, we must prove that it cannot be transformed into the line element (49) nor into one which differs from it by a constant factor. In our investigation we will make use of the well known criteria for the transformability of two differential quadratic forms established by Christoffel and Lipshitz, but most of all we utilize here the circumstance that the two forms to be compared admit two respective 4-parameter groups G_4, Γ_4 of transformations into themselves, making available for us the general theorems of Lie. Therefore we make the following observation that we will equally apply to the analogous research of the following sections. The supposed equations of transformation

$$x_1 = \varphi_1(y_1, y_2, y_3) \ , \ x_2 = \varphi_2(y_1, y_2, y_3) \ , \ x_3 = \varphi_3(y_1, y_2, y_3)$$

must obviously transform the group of motions G_4 of the one space into the Γ_4 of the other. First it is necessary to see if the two groups G_4, Γ_4 are similar. When this *necessary* condition is satisfied the assumed transformability of the two line elements still does not follow from it, but there will remain only to see if the equation found by transforming G_4 into Γ_4 can be specialized so that it also puts the two spaces into the relation of similarity.

To see if the two groups G_4, Γ_4 are similar, according to the general criteria of Lie[37] we must first of all get the groups into an isomorphic correspondence[38] in the most general way. Therefore with (§17)

$$Y_1f = \frac{\partial f}{\partial y_2} \ , \ Y_2f = \frac{\partial f}{\partial y_3} \ , \ Y_3f = \frac{\partial f}{\partial y_1} - y_2\frac{\partial f}{\partial y_2} \ ,$$
$$Y_4f = y_2\frac{\partial f}{\partial y_1} + \frac{1}{2}\left(\frac{e^{-2y_1}}{1-m^2} - y_2^2\right)\frac{\partial f}{\partial y_2} - \frac{me^{-y_1}}{1-m^2}\frac{\partial f}{\partial y_3} \ ,$$

[36] See the preface.
[37] S. Lie-F. Engel, Vol. I, p. 327.
[38] In Italian: “isomorfismo oleodrico” [Translator].

as the generating transformations of Γ_4, with the same composition as the generators X_1f, X_2f, X_3f, X_4f of G_4, it will be useful to choose in G_4 (in the most general way) four other generators $\bar{X}_1f, \bar{X}_2f, \bar{X}_3f, \bar{X}_4f$ so that they still have the same composition, namely one has:

$$[\bar{X}_1, \bar{X}_2]f = 0 \ , \ [\bar{X}_1, \bar{X}_3]f = -\bar{X}_1f \ , \ [\bar{X}_1, \bar{X}_4]f = \bar{X}_3f \ ,$$
$$[\bar{X}_2, \bar{X}_3]f = 0 \ , \ [\bar{X}_2, \bar{X}_4]f = 0 \ , \ [\bar{X}_3, \bar{X}_4]f = -\bar{X}_4f \ .$$

If one observes first that the derived group of G_4 coincides either with $(\bar{X}_1f, \bar{X}_3f, \bar{X}_4f)$ or with (X_1f, X_3f, X_4f), it follows from this that X_1f, X_3f, X_4f must be composed of only $\bar{X}_1f, \bar{X}_3f, \bar{X}_4f$. Moreover since $\bar{X}_2f$, like X_2f, is the only transformation in G_4 which commutes with every other in the group, X_2f must differ from $\bar{X}_2f$ by a constant factor λ; we have therefore

$$\begin{aligned} X_1f &= \alpha_1\bar{X}_1f + \alpha_2\bar{X}_3f + \alpha_3\bar{X}_4f \ , \\ X_3f &= \beta_1\bar{X}_1f + \beta_2\bar{X}_3f + \beta_3\bar{X}_4f \ , \\ X_4f &= \gamma_1\bar{X}_1f + \gamma_2\bar{X}_3f + \gamma_3\bar{X}_4f \ , \\ X_2f &= \lambda\bar{X}_2f \ , \end{aligned} \tag{56}$$

with $\alpha, \beta, \gamma, \lambda$ being constants. The composition relations translate into the following equations for α, β, γ; the α, γ must be constrained by the relations

$$\alpha_2^2 + 2\alpha_1\alpha_3 = 0 \ , \ \gamma_2^2 + 2\gamma_1\gamma_3 = 0 \ , \ \alpha_1\gamma_3 + \alpha_3\gamma_1 + \alpha_2\gamma_2 = 1 \tag{57}$$

and the β must be expressed in terms of these by the formulas

$$\beta_1 = \alpha_2\gamma_1 - \alpha_1\gamma_2 \ , \ \beta_2 = \alpha_1\gamma_3 - \alpha_3\gamma_1 \ , \ \beta_3 = \alpha_3\gamma_2 - \alpha_2\gamma_3 \ . \tag{58}$$

In order to check what follows, it is worth noting that the following relations are a consequence of the ones above

$$\begin{aligned} &\alpha_3\beta_1 + \alpha_1\beta_3 + \alpha_2\beta_2 = 0 \ , \ \gamma_3\beta_1 + \gamma_1\beta_3 + \gamma_2\beta_2 = 0 \ , \\ &\beta_2^2 + 2\beta_1\beta_3 = 1 \ . \end{aligned} \tag{59}$$

As a consequence the determinant $\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}$ is equal to $+1$, and solving for the $\bar{X}f$ one has[39]

$$\begin{aligned} \bar{X}_1f &= \gamma_3X_1f + \beta_3X_3f + \alpha_3X_4f \ , \\ \bar{X}_3f &= \gamma_2X_1f + \beta_2X_3f + \alpha_2X_4f \ , \\ \bar{X}_4f &= \gamma_1X_1f + \beta_1X_3f + \alpha_1X_4f \ , \end{aligned}$$

by which all the minors of second order of this determinant are equal to one element, for example, $\alpha_3 = \alpha_2\beta_3 - \alpha_3\beta_2$, etc.

[39]The second and third equation in the original paper had $\bar{X}_2f$ and $\bar{X}_3f$ on the l.h.s., respectively; correction based on the *Opere* [Editor].

So having put the two groups $G_4 \equiv (\bar{X}_1 f, \bar{X}_2 f, \bar{X}_3 f, \bar{X}_4 f)$, $\Gamma_4 \equiv (Y_1 f,\ Y_2 f,\ Y_3 f,\ Y_4 f)$ into an isomorphic correspondence, it is necessary to identify the relationship among $Y_1 f, Y_2 f, Y_3 f, Y_4 f$, namely:

$$Y_4 f = \frac{1}{2}\left(\frac{e^{-2y_1}}{1-m^2} + y_2^2\right) Y_1 f - \frac{m e^{-y_1}}{1-m^2} Y_2 f + y_2 Y_3 f \tag{60}$$

with the one which correspondingly relates $\bar{X}_4 f$ to $\bar{X}_1 f, \bar{X}_2 f, \bar{X}_3 f$. Now substituting into the relation

$$X_4 f = \frac{1}{2}\left(\frac{e^{-2x_1}}{1-n^2} + x_2^2\right) X_1 f - \frac{n e^{-x_1}}{1-n^2} X_2 f + x_2 X_3 f$$

the values (56), we find:[40]

$$\begin{aligned} &\left\{\frac{\alpha_3}{2}\left(\frac{e^{-2x_1}}{1-n^2} + x_2^2\right) + \beta_3 x_2 - \gamma_3\right\} \bar{X}_4 f \\ &= \left\{\gamma_1 - \beta_1 x_2 - \frac{\alpha_1}{2}\left(\frac{e^{-2x_1}}{1-n^2} + x_2^2\right)\right\} \bar{X}_1 f \\ &\quad + \left\{\gamma_2 - \beta_2 x_2 - \frac{\alpha_2}{2}\left(\frac{e^{-2x_1}}{1-n^2} + x_2^2\right)\right\} \bar{X}_3 f + \frac{n\lambda e^{-x_1}}{1-n^2} \bar{X}_2 f\ . \end{aligned} \tag{60*}$$

We introduce the abbreviations

$$\frac{e^{-x_1}}{\sqrt{1-n^2}} = \xi\ ,\quad \frac{e^{-y_1}}{\sqrt{1-m^2}} = \eta\ ,\quad \lambda\frac{n\sqrt{1-m^2}}{m\sqrt{1-n^2}} = \mu \tag{61}$$

and identifying the coefficients of (60), (60*) we find the three equations[41]

$$y_2 = \frac{\gamma_2 - \beta_2 x_2 - \alpha_2(\xi^2 + x_2^2)/2}{\alpha_3(\xi^2 + x_2^2)/2 + \beta_3 x_2 - \gamma_3}\ , \tag{62a}$$

$$\eta = \frac{-\mu\xi}{\alpha_3(\xi^2 + x_2^2)/2 + \beta_3 x_2 - \gamma_3}\ , \tag{62b}$$

$$\begin{aligned} &\frac{1}{2}(\eta^2 + y_2^2)\left\{\alpha_3(\xi^2 + x_2^2)/2 + \beta_3 x_2 - \gamma_3\right\} \\ &\quad + \frac{\alpha_1}{2}(\xi^2 + x_2^2) + \beta_1 x_2 - \gamma_1 = 0\ . \end{aligned} \tag{62*}$$

If these three equations are compatible the two groups are similar and equations (62) then give in the corresponding equations of transformation y_1, y_2 expressed in terms of x_1, x_2 (Lie, *ibid.*). Now by substituting the values of y_2, η given by (62) into (62*) and completing the square, we obtain

$$\begin{aligned} &\mu^2\xi^2 + \mu^2\left[\frac{\alpha_2}{2}\xi^2 + \frac{\alpha_2}{2}x_2^2 + \beta_2 x_2 - \gamma_2\right]^2 \\ &+2\left(\frac{\alpha_3}{2}\xi^2 + \frac{\alpha_3}{2}x_2^2 + \beta_3 x_2 - \gamma_3\right)\left(\frac{\alpha_1}{2}\xi^2 + \frac{\alpha_1}{2}x_2^2 + \beta_1 x_2 - \gamma_1\right) = 0 \end{aligned}$$

[40]The last term on the r.h.s. was preceded by a minus sign in the original, now corrected to a positive sign after the *Opere* [Editor].

[41]The r.h.s. of (62b) lacked the minus sign in the original, now corrected after the *Opere* [Editor].

which must therefore be an identity in ξ, x_2. Taking into account the relations (57), (58), (59) among the constants α, β, γ one immediately finds it necessary and sufficient for this to be true that one have $\mu^2 = 1$, namely $\lambda^2 = \frac{m^2(1-n^2)}{n^2(1-m^2)}$.

One concludes from this that the two groups G_4, Γ_4 are indeed similar and for the most general equations which transform the one group into the other, one necessarily has[42]

$$y_2 = \frac{\gamma_2 - \beta_2 x_2 - \alpha_2(\xi^2 + x_2^2)/2}{\alpha_3(\xi^2 + x_2^2)/2 + \beta_3 x_2 - \gamma_3} \ , \tag{63a}$$

$$\eta = \frac{\pm\xi}{\alpha_3(\xi^2 + x_2^2)/2 + \beta_3 x_2 - \gamma_3} \ , \tag{63b}$$

from which, as one sees, y_1, y_2 are independent of x_3.

19 The constant n is essential in $ds^2 = dx_1^2 + e^{2x_1}\, dx_2^2 + 2ne^{x_1}\, dx_2 dx_3 + dx_3^2$.

To demonstrate this claim we observe finally that since the equations of transformation must change $X_1 f, X_2 f, X_3 f, X_4 f$ respectively into

$$\alpha_1 Y_1 f + \alpha_2 Y_3 f + \alpha_3 Y_4 f \ , \ \lambda Y_2 f \ ,$$
$$\beta_1 Y_1 f + \beta_2 Y_3 f + \beta_3 Y_4 f \ , \ \gamma_1 Y_1 f + \gamma_2 Y_3 f + \gamma_3 Y_4 f \ ,$$

from these follow the values of all the first partial derivatives of the y with respect to the x.[43] Of these equations it is enough for us to write the following ones:

$$\frac{\partial y_1}{\partial x_2} = \alpha_2 + \alpha_3 y_2 \ ,$$

$$\frac{\partial y_3}{\partial x_1} = -\frac{me^{-y_1}}{1-m^2}(\alpha_3 x_2 + \beta_3) \ , \ \frac{\partial y_3}{\partial x_2} = -\frac{m\alpha_3 e^{-y_1}}{1-m^2} \ , \ \frac{\partial y_3}{\partial x_3} = \lambda \ . \tag{64}$$

By substituting the expression (63) for y_2 into the value of $\partial y_1/\partial x_2$ one has

$$\frac{\partial y_1}{\partial x_2} = \frac{\alpha_3 x_2 + \beta_3}{\alpha_3(\xi^2 + x_2^2)/2 + \beta_3 x_2 - \gamma_3} \ . \tag{65}$$

Given this, from the assumed transformability of the two line elements we will have:

$$\begin{aligned} & dy_1^2 + e^{2y_1}\, dy_2^2 + 2me^{y_1}\, dy_2 dy_3 + dy_3^2 \\ = \ & \lambda^2\{dx_1^2 + e^{2x_1}\, dx_2^2 + 2ne^{x_1} dx_2 dx_3 + dx_3^2\} \ . \end{aligned} \tag{66}$$

We now apply the equations of Christoffel

$$\frac{\partial^2 y_\nu}{\partial x_r \partial x_s} + \sum_{i,k} \left\{ \begin{matrix} \nu \\ ik \end{matrix} \right\}_y \frac{\partial y_i}{\partial x_r}\frac{\partial y_k}{\partial x_s} = \sum_\mu \left\{ \begin{matrix} \mu \\ rs \end{matrix} \right\}_x \frac{\partial y_\nu}{\partial x_\mu} \ , \ (\nu, r, s = 1, 2, 3) \ ,$$

[42] The original paper had α_2 instead of α_3 in (63a), which was a typo [Editor].

[43] In general $X_i f$ is changed into $\partial f/\partial y_1\, X_i(y_1) + \partial f/\partial y_2\, X_i(y_2) + \partial f/\partial y_3\, X_i(y_3)$, hence the formulas indicated in the text.

the index x or y attached to the Christoffel symbol indicating whether it is constructed in terms of the form of the x or that of the y.[44] Setting $\nu = 2$, $r = 2$, $s = 3$ and replacing the Christoffel symbols by their actual values one obtains $\frac{m^2\lambda}{1-m^2}\partial y_1/\partial x_2 = ne^{x_1}\partial y_3/\partial x_1$, or using (64), (65)

$$\frac{m^2\lambda e^{-x_1}}{1-m^2}\frac{\alpha_3 x_2 + \beta_3}{\alpha_3(\xi^2 + x_2^2)/2 + \beta_3 x_2 - \gamma_3} + \frac{nme^{-y_1}}{1-m^2}(\alpha_3 x_2 + \beta_3) = 0 \ .$$

Therefore, if $\alpha_3 = 0$, $\beta_3 = 0$ does not hold, one must have

$$m\lambda\sqrt{1-n^2}\xi + n\sqrt{1-m^2}\eta\,\{\alpha_3(\xi^2 + x_2^2)/2 + \beta_3 x_2 - \gamma_3\} = 0 \ ,$$

namely by (62b)

$$m\lambda\sqrt{1-n^2} - n\sqrt{1-m^2}\mu = 0 \ ,$$

or equivalently

$$m^2(1-n^2) - n^2(1-m^2) = 0 \ ,$$

or equivalently

$$m^2(1-n^2) + n^2(1-m^2) = 0 \ ,$$

which is absurd since $n^2 < 1$, $m^2 < 1$. Therefore we will have $\alpha_3 = \beta_3 = 0$ implying $\alpha_2 = 0$, $\beta_2^2 = 1$, from which (63) tells us that y_2 is only a function of x_2, and y_1 differs from x_1 only by an additive constant. After this (66) immediately gives $\lambda^2 = 1$ (comparing the terms in dx_1^2), namely

$$m^2(1-n^2) = n^2(1-m^2) \ ,$$

and consequently $n^2 = m^2$ as indeed we wished to show.[45]

20 The groups of type IV: $[X_1, X_2]f = 0$, $[X_1, X_3]f = X_1 f$, $[X_2, X_3]f = X_1 f + X_2 f$.

To apply the equations of §14 to the present composition we must set $a = 1$, $b = 0$, $c = 0$, $a' = 1$, $b' = 1$, $c' = 0$. From this it follows that ξ_1 is constant, so we set $\xi_1 = -2/h$ and the last three of (F) §14 become

$$\alpha' = h\alpha \ , \ \beta' = h(\alpha + 2\beta)/2 \ , \ \gamma' = h(\beta + \gamma) \ ,$$

[44]It is useful to note that the Christoffel symbols of the second kind $\left\{ {l \atop ik} \right\}$ are not changed in value by multiplying the line element by a constant factor.

[45]The signs preceding the second terms on the l.h.s. in the previous two displayed equations were plus signs in the original (propagating from sign errors noted above), requiring a further short argument to obtain the desired result now deleted after the *Opere*. The deleted material after the second displayed equation was:

> "which is absurd since $n^2 < 1$, $m^2 < 1$. Therefore we will have $\alpha_3 = \beta_3 = 0$ implying $\alpha_2 = 0$, $\beta_2^2 = 1$, from which (63) tells us that y_2 is only a function of x_2, and y_1 differs from x_1 only by an additive constant. After this (66) immediately gives $\lambda^2 = 1$ (comparing the terms in dx_1^2), namely
>
> $$m^2(1-n^2) = n^2(1-m^2) \ ,"$$

[Editor].

from which by integrating and conveniently disposing of a factor independent of x_2, x_3, we can assume[46]

$$\alpha = e^{hx_1}\ ,\ \beta = e^{hx_1}(hx_1/2 + l)\ ,\ \gamma = e^{hx_1}[(hx_1/2 + l)^2 + m^2]\ ,$$

with l, m constants. Changing x_1 into $x_1 + constant$ and replacing the space with a similar space, we have for the standard form of the present case

$$ds^2 = dx_1{}^2 + e^{x_1}\{dx_2^2 + 2x_1\, dx_2 dx_3 + (x_1^2 + n^2)\, dx_3^2\}\ . \tag{67}$$

Applying equations (E) §14 to determine the most general infinitesimal motion of the space, we find the following equations:

$$\frac{\partial \eta_1}{\partial x_1} = 0\ , \tag{68}$$

$$e^{-x_1}\frac{\partial \eta_1}{\partial x_2} + \frac{\partial \eta_2}{\partial x_1} + x_1\frac{\partial \eta_3}{\partial x_1} = 0\ , \tag{69a}$$

$$e^{-x_1}\frac{\partial \eta_1}{\partial x_3} + x_1\frac{\partial \eta_2}{\partial x_1} + (x_1{}^2 + n^2)\frac{\partial \eta_3}{\partial x_1} = 0\ , \tag{69b}$$

$$\frac{1}{2}\eta_1 + \frac{\partial \eta_2}{\partial x_2} + x_1\frac{\partial \eta_3}{\partial x_2} = 0\ , \tag{70}$$

$$\frac{1}{2}(x_1^2 + n^2 + 2x_1)\eta_1 + x_1\frac{\partial \eta_2}{\partial x_3} + (x_1^2 + n^2)\frac{\partial \eta_3}{\partial x_3} = 0\ , \tag{71}$$

$$(x_1 + 1)\eta_1 + \frac{\partial \eta_2}{\partial x_3} + x_1\left(\frac{\partial \eta_2}{\partial x_2} + \frac{\partial \eta_3}{\partial x_3}\right) + (x_1^2 + n^2)\frac{\partial \eta_3}{\partial x_2} = 0\ . \tag{72}$$

Solving (69) we obtain

$$\frac{\partial \eta_2}{\partial x_1} = \frac{e^{-x_1}}{n^2}\left[x_1\frac{\partial \eta_1}{\partial x_3} - (x_1^2 + n^2)\frac{\partial \eta_1}{\partial x_2}\right]\ ,$$
$$\frac{\partial \eta_3}{\partial x_1} = \frac{e^{-x_1}}{n^2}\left[x_1\frac{\partial \eta_1}{\partial x_2} - \frac{\partial \eta_1}{\partial x_3}\right]\ ,$$

and integrating with respect to x_1, of which η_1 is independent by (68), we have

$$\eta_2 = \frac{e^{-x_1}}{n^2}\left[(x_1^2 + n^2 + 2x_1 + 2)\frac{\partial \eta_1}{\partial x_2} - (x_1 + 1)\frac{\partial \eta_1}{\partial x_3}\right] + \psi(x_2, x_3)\ ,$$
$$\eta_3 = \frac{e^{-x_1}}{n^2}\left[\frac{\partial \eta_1}{\partial x_3} - (x_1 + 1)\frac{\partial \eta_1}{\partial x_2}\right] + \chi(x_2, x_3)\ .$$

Substituting into (70), we immediately deduce from this

$$\frac{\partial^2 \eta_1}{\partial x_2{}^2} = \frac{\partial^2 \eta_1}{\partial x_2 \partial x_3} = 0\ ,\ \frac{\partial \chi}{\partial x_2} = 0\ ,\ \frac{\partial \psi}{\partial x_2} = -\frac{1}{2}\eta_1\ ,$$

so that the substitution in (71) gives

$$\frac{\partial^2 \eta_1}{\partial x_3{}^2} = 0\ ,\ \frac{\partial \chi}{\partial x_3} = -\frac{1}{2}\eta_1\ ,\ \frac{\partial \psi}{\partial x_3} = -\eta_1\ ,$$

[46] hx_1 in the original paper was corrected to $hx_1/2$ after the *Opere* [Editor].

hence

$$\frac{\partial \eta_1}{\partial x_2} = -2\frac{\partial^2 \chi}{\partial x_2 \partial x_3} = 0 \ , \ \frac{\partial \eta_1}{\partial x_3} = -2\frac{\partial^2 \psi}{\partial x_2 \partial x_3} = 2\frac{\partial \eta_1}{\partial x_2} = 0 \ .$$

In this way we find for the most general values of η_1, η_2, η_3 the formulas

$$\eta_1 = a \ , \ \eta_2 = -\frac{a}{2}x_2 - ax_3 + b \ , \ \eta_3 = -\frac{a}{2}x_3 + c \ ,$$

with a, b, c arbitrary constants. Therefore the complete group of motions is only a G_3 here; it is generated by the 3 infinitesimal transformations

$$X_1 f = \frac{\partial f}{\partial x_2} \ , \ X_2 f = \frac{\partial f}{\partial x_3} \ ,$$
$$X_3 f = -\frac{\partial f}{\partial x_1} + \left(\frac{x_2}{2} + x_3\right)\frac{\partial f}{\partial x_2} + \frac{x_3}{2}\frac{\partial f}{\partial x_3} \tag{73}$$

and has the composition[47]

$$[X_1, X_2]f = 0 \ , \ [X_1, X_3]f = \frac{1}{2}X_1 f \ , \ [X_2, X_3]f = X_1 f + \frac{1}{2}X_2 f \ .$$

21 The constant n is essential in $ds^2 = dx_1^2 + e^{x_1}[dx_2^2 + 2x_1\, dx_2 dx_3 + (x_1^2 + n^2)\, dx_3^2]$.

Analogously to what we have done for the spaces of §17, we also want to see here if the constant n of the present line element (67) is essential. We respond affirmatively to the question by showing that a second line element[48]

$$ds^2 = dy_1^2 + e^{y_1}\,[dy_2^2 + 2y_1\, dy_2 dy_3 + (y_1^2 + m^2)\, dy_3^2] \ , \tag{74}$$

where $m^2 \neq n^2$, cannot be identified with the original nor be proportional to it. Proceeding exactly as in §18 we first compare the two respective groups of motions G_3, Γ_3, the first generated by the transformations (73), the second instead by[49]

$$Y_1 f = \frac{\partial f}{\partial y_2} \ , \ Y_2 f = \frac{\partial f}{\partial y_3} \ , \ Y_3 f = -\frac{\partial f}{\partial y_1} + \left(\frac{y_2}{2} + y_3\right)\frac{\partial f}{\partial y_2} + \frac{y_3}{2}\frac{\partial f}{\partial y_3}$$

with the same composition

$$[Y_1, Y_2]f = 0 \ , \ [Y_1, Y_3]f = \frac{1}{2}Y_1 f \ , \ [Y_2, Y_3]f = Y_1 f + \frac{1}{2}Y_2 f \ .$$

We must find the most general transformation which changes the one group into the other and see if it can give rise to the hypothesized transformation of the two line elements. We therefore take three other transformation generators of Γ_3, let them be $\bar{Y}_1 f, \bar{Y}_2 f, \bar{Y}_3 f$, which have the same composition as above; we therefore have

$$\bar{Y}_1 f = \alpha Y_1 f + \beta Y_2 f \ , \bar{Y}_2 f = \gamma Y_1 f + \delta Y_2 f \ ,$$
$$\bar{Y}_3 f = aY_1 f + bY_2 f + cY_3 f \ ,$$

[47] To have the canonical composition it would suffice to double $X_1 f, X_3 f$.

[48] Equation number "(74)" is missing in the original, added by the Editor.

[49] The second term on the r.h.s. of the 3rd equation had $\partial f/\partial y_3$ in the original, which was incorrect [Editor].

and among the constants $\alpha, \beta, \gamma, \delta, c$ the relations

$$\alpha c + 2\beta c = \alpha \ , \ \beta c = \beta \ , \ \gamma c + 2\delta c = 2\alpha + \gamma \ , \ \delta c = 2\beta + \delta \ ,$$

from which it follows that $\beta = 0$, $c = 1$, $\delta = \alpha$ and consequently:

$$\begin{aligned} &\bar{Y}_1 f = \alpha Y_1 f \ , \bar{Y}_2 f = \gamma Y_1 f + \alpha Y_2 f \ , \\ &\bar{Y}_3 f = a Y_1 f + b Y_2 f + Y_3 f \ . \end{aligned}$$

There certainly exist transformations which change $X_1 f$, $X_2 f$, $X_3 f$ respectively into $\bar{Y}_1 f, \bar{Y}_2 f, \bar{Y}_3 f$ because the two simply transitive groups are isomorphic.[50] For one such transformation the partial derivatives of the y with respect to the x must assume the following values:

$$\begin{aligned} &\frac{\partial y_1}{\partial x_1} = 1 \ , \ \frac{\partial y_1}{\partial x_2} = 0 \ , \ \frac{\partial y_1}{\partial x_3} = 0 \ , \\ &\frac{\partial y_2}{\partial x_1} = \alpha\left(\frac{x_2}{2} + x_3\right) + \gamma\frac{x_3}{2} - a - \frac{y_2}{2} - y_3 \ , \ \frac{\partial y_2}{\partial x_2} = \alpha \ , \ \frac{\partial y_2}{\partial x_3} = \gamma \ , \\ &\frac{\partial y_3}{\partial x_1} = \alpha\frac{x_3}{2} - b - \frac{y_3}{2} \ , \ \frac{\partial y_3}{\partial x_2} = 0 \ , \ \frac{\partial y_3}{\partial x_3} = \alpha \ . \end{aligned}$$

Integrating we have the actual equations in finite terms

$$\begin{aligned} &y_1 = x_1 + h \ , \\ &y_2 = \alpha x_2 + \gamma x_3 + k e^{-x_1/2} - c x_1 e^{-x_1/2} - 2a + 4b \ , \\ &y_3 = \alpha x_3 + c e^{-x_1/2} - 2b \ , \end{aligned}$$

with h, c, k new constants. The line element (74) therefore becomes:

$$\begin{aligned} &dx_1^2 + e^h \left\{\left(\frac{c x_1}{2} - c - \frac{k}{2}\right) dx_1 + \alpha e^{\frac{x_1}{2}} dx_2 + \gamma e^{\frac{x_1}{2}} dx_3\right\}^2 \\ &+2e^h (x_1 + h)\left\{\left(\frac{c x_1}{2} - c - \frac{k}{2}\right) dx_1 + \alpha e^{\frac{x_1}{2}} dx_2 + \gamma e^{\frac{x_1}{2}} dx_3\right\} \\ &\qquad \times \left\{-\frac{c}{2} dx_1 + \alpha e^{\frac{x_1}{2}} dx_3\right\} \\ &+e^h \left\{(x_1 + h)^2 + m^2\right\} \cdot \left\{-\frac{c}{2} dx_1 + \alpha e^{\frac{x_1}{2}} dx_3\right\}^2 \ . \end{aligned}$$

Comparing with the line element (67) we must set the coefficients of $dx_1 dx_2$ and $dx_1 dx_3$ to zero; we immediately find $c = 0$, $k = 0$; and then comparing the terms in dx_2^2, $dx_2 dx_3$ and dx_3^2 one deduces [51]

$$\begin{aligned} &\alpha^2 e^h = 1 \ , \ \{\gamma\alpha + \alpha^2(x_1 + h)\} e^h = x_1 \ , \\ &\{\gamma^2 + 2\alpha\gamma(x_1 + h) + \alpha^2(x_1 + h)^2 + \alpha^2 m^2\} e^h = x_1^2 + n^2 \ . \end{aligned}$$

From this it follows that $\alpha^2 e^h = 1$, $\gamma + \alpha h = 0$, so that $n^2 = m^2$, Q.E.D.

[50] S. Lie-F. Engel, Vol. I, p. 340.

[51] This partial sentence is the translator's interpretation of Bianchi's intended meaning. Bianchi's original phrase in which he meant to refer to equation (67) "Comparing this with the line element (64), we must set the coefficients of $dx_1 dx_2$ and $dx_1 dx_3$ to zero, which leads to..." has a proof correction at the end of his article (implemented by the Editors of *Opere*) stating "The penultimate line on p. 312 should read: equating the coefficients of $dx_1 dx_2$ and $dx_1 dx_3$ we find immediately", but this omits the necessary "to zero" and removes any equation number [Editor].

22 The groups of type V:

$[X_1, X_2]f = 0\ ,\ [X_1, X_3]f = X_1 f,\ [X_2, X_3]f = X_2 f.$

The constants a, b, c, a', b', c' of §14 here take the values $a = 1$, $b = 0$, $c = 0$, $a' = 0$, $b' = 1$, $c' = 0$, from which it follows that ξ_1 is again constant, so we set $\xi_1 = -2/h$, and the last three of (F) §14 give

$$\alpha' = h\alpha\ ,\ \gamma' = h\gamma\ ,\ \beta' = h\beta\ ,$$

from which by integrating we have

$$\alpha = le^{hx_1}\ ,\ \beta = me^{hx_1}\ ,\ \gamma = ne^{hx_1}\ ,$$

with l, m, n constants. Changing (linearly) the parameters x_2, x_3 we obtain

$$ds^2 = dx_1^2 + e^{2hx_1}\,(dx_2^2 + dx_3^2)\ ,$$

the line element which belongs to the space of constant negative curvature.

In this case the existence of the transitive group G_3 of motions of the designated type implies a complete group of motions (non-Euclidean) of 6 parameters.

23 The groups of type VI:

$[X_1, X_2]f = 0\ ,\ [X_1, X_3]f = X_1 f,\ [X_2, X_3]f = hX_2 f\ ,\ h \neq 0, 1.$

For the groups of this type we must set $a = 1$, $b = c = 0$, $a' = 0$, $b' = h$, $c' = 0$ in the equations. From this it follows that ξ_1 is constant, so we set $\xi_1 = -2/k$ and the usual equations (F) §14 give us

$$\alpha' = k\alpha\ ,\ 2\beta' = k(h+1)\beta\ ,\ \gamma' = hk\gamma\ ,$$

from which by integrating and absorbing two of the constants of integration into x_2, x_3, we find

$$\alpha = e^{kx_1}\ ,\ \beta = ne^{k(h+1)x_1/2}\ ,\ \gamma = e^{hkx_1}\ ,$$

where n is a constant which can clearly be assumed positive, so that (because $\alpha\gamma - \beta^2 > 0$) we will have $0 < n < 1$.

By passing to a similar space we can make $k = 2$, so

$$ds^2 = dx_1^2 + e^{2x_1}\,dx_2^2 + 2ne^{(h+1)x_1}\,dx_2 dx_3 + e^{2hx_1}\,dx_3^2\ . \tag{76}$$

One will observe that for $h = 0$ this reduces to type III and the line element (76) is then changed into the one (49) of the spaces of §17. [52]

The most general infinitesimal motion of this space is determined, according to (E) §14, by the following equations:[53]

$$\frac{\partial \eta_1}{\partial x_1} = 0\ , \tag{77}$$

[52] We have not been able to treat the particular case $h = 0$ together with the general case, because only for $h = 0$ does one have a 4-parameter group of motions.

[53] The factor n in the last term of (79) is absent in the original, correction after the *Opere* [Editor].

$$\frac{\partial\eta_1}{\partial x_2} + e^{2x_1}\frac{\partial\eta_2}{\partial x_1} + ne^{(h+1)x_1}\frac{\partial\eta_3}{\partial x_1} = 0 \ , \tag{78a}$$

$$\frac{\partial\eta_1}{\partial x_3} + ne^{(h+1)x_1}\frac{\partial\eta_2}{\partial x_1} + e^{2hx_1}\frac{\partial\eta_3}{\partial x_1} = 0 \ , \tag{78b}$$

$$e^{x_1}\left(\eta_1 + \frac{\partial\eta_2}{\partial x_2}\right) + ne^{hx_1}\frac{\partial\eta_3}{\partial x_2} = 0 \ , \tag{79}$$

$$h\eta_1 + ne^{(1-h)x_1}\frac{\partial\eta_2}{\partial x_3} + \frac{\partial\eta_3}{\partial x_3} = 0 \ , \tag{80}$$

$$\begin{aligned} &n(h+1)e^{(h+1)x_1}\eta_1 + e^{2x_1}\frac{\partial\eta_2}{\partial x_3} + ne^{(h+1)x_1}\left(\frac{\partial\eta_2}{\partial x_2} + \frac{\partial\eta_3}{\partial x_3}\right) \\ &\quad + e^{2hx_1}\frac{\partial\eta_3}{\partial x_2} = 0 \ . \end{aligned} \tag{81}$$

Solving (78) for $\partial\eta_2/\partial x_1$, $\partial\eta_3/\partial x_1$, we obtain

$$\begin{aligned} \frac{\partial\eta_2}{\partial x_1} &= \frac{ne^{-(h+1)x_1}}{1-n^2}\frac{\partial\eta_1}{\partial x_3} - \frac{e^{-2x_1}}{1-n^2}\frac{\partial\eta_1}{\partial x_2} \ , \\ \frac{\partial\eta_3}{\partial x_1} &= \frac{ne^{-(h+1)x_1}}{1-n^2}\frac{\partial\eta_1}{\partial x_2} - \frac{e^{-2hx_1}}{1-n^2}\frac{\partial\eta_1}{\partial x_3} \ . \end{aligned} \tag{82}$$

The integration of these last two with respect to x_1 leads us to separate the two cases

$$a)\ h = -1 \ , \qquad b)\ h \neq -1 \ .$$

In case a) by integrating we obtain

$$\begin{aligned} \eta_2 &= \frac{nx_1}{1-n^2}\frac{\partial\eta_1}{\partial x_3} + \frac{e^{-2x_1}}{2(1-n^2)}\frac{\partial\eta_1}{\partial x_2} + \psi(x_2,x_3) \ , \\ \eta_3 &= \frac{nx_1}{1-n^2}\frac{\partial\eta_1}{\partial x_2} - \frac{e^{2x_1}}{2(1-n^2)}\frac{\partial\eta_1}{\partial x_3} + \chi(x_2,x_3) \ , \end{aligned}$$

and substituting into the successive equations (79), (80), (81), we find

$$\begin{aligned} &\frac{\partial^2\eta_1}{\partial {x_2}^2} = \frac{\partial^2\eta_1}{\partial x_2\partial x_3} = \frac{\partial^2\eta_1}{\partial {x_3}^2} = 0 \ , \\ &\frac{\partial\psi}{\partial x_3} = 0 \ , \ \frac{\partial\psi}{\partial x_2} = -\eta_1 \ , \ \frac{\partial\chi}{\partial x_2} = 0 \ , \ \frac{\partial\chi}{\partial x_3} = \eta_1 \ . \end{aligned}$$

from which it follows that

$$\eta_1 = a \ , \ \eta_2 = -ax_2 + b \ , \ \eta_3 = ax_3 + c \ ,$$

with a, b, c constants. In this case therefore the group of motions is only a G_3 with the infinitesimal transformation generators:

$$X_1 f = \frac{\partial f}{\partial x_2} \ , \ X_2 f = \frac{\partial f}{\partial x_3} \ , \ X_3 f = -\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} - x_3\frac{\partial f}{\partial x_3} \ .$$

Case b). Now let $h \neq -1$. Equations (82) integrated give

$$\begin{aligned} \eta_2 &= -\frac{ne^{(h+1)x_1}}{(h+1)(1-n^2)}\frac{\partial\eta_1}{\partial x_3} + \frac{e^{-2x_1}}{2(1-n^2)}\frac{\partial\eta_1}{\partial x_2} + \psi(x_2,x_3) \ , \\ \eta_3 &= -\frac{ne^{(h+1)x_1}}{(h+1)(1-n^2)}\frac{\partial\eta_1}{\partial x_2} + \frac{e^{-2hx_1}}{2h(1-n^2)}\frac{\partial\eta_1}{\partial x_3} + \chi(x_2,x_3) \ , \end{aligned}$$

and substituting into (79) remembering that h is different from $0, 1, -1$, we find

$$\frac{\partial^2 \eta_1}{\partial {x_2}^2} = 0 \ , \ n\frac{\partial^2 \eta_1}{\partial x_2 \partial x_3} = 0 \ , \ \frac{\partial \chi}{\partial x_2} = 0 \ , \ \frac{\partial \psi}{\partial x_2} = -\eta_1 \ .$$

Substituting into (80) therefore gives

$$\frac{\partial^2 \eta_1}{\partial {x_3}^2} = 0 \ , \ \frac{\partial \psi}{\partial x_3} = 0 \ , \ \frac{\partial \chi}{\partial x_3} = -h\eta_1 \ ,$$

and from this follow for η_1, η_2, η_3 the values

$$\eta_1 = a \ , \ \eta_2 = -ax_2 + b \ , \ \eta_3 = -hax_3 + c \ ,$$

with a, b, c arbitrary constants. We therefore have as the complete group of motions the G_3 generated by the three infinitesimal transformations

$$X_1 f = \frac{\partial f}{\partial x_2} \ , \ X_2 f = \frac{\partial f}{\partial x_3} \ , \ X_3 f = -\frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + hx_3 \frac{\partial f}{\partial x_3} \ .$$

which indeed has the composition

$$[X_1, X_2]f = 0 \ , [X_1, X_3]f = X_1 f \ , [X_2, X_3]f = hX_2 f \ .$$

We now see that the result obtained above for $h = -1$ is included in the general case.

24 The constant n is essential in[54] $ds^2 = dx_1^2 + e^{2x_1}dx_2^2 + 2ne^{(h+1)x_1}dx_2 dx_3 + e^{2hx_1}dx_3^2$.

That the constant h is essential in this line element is clear since it is already essential in the composition of its group of motions; but now we wish to show that the constant n (apart from sign) is also essential. Therefore let there be the two spaces

$$ds^2 = dx_1^2 + e^{2x_1}dx_2^2 + 2ne^{(h+1)x_1}dx_2 dx_3 + e^{2hx_1}dx_3^2 \qquad (\alpha)$$
$$ds^2 = dy_1^2 + e^{2y_1}dy_2^2 + 2me^{(h+1)y_1}dy_2 dy_3 + e^{2hy_1}dy_3^2 \ ; \qquad (\beta)$$

we wish to prove that assuming the two spaces are similar implies $n^2 = m^2$.

The group Γ_3 of motions of the second space is generated by the three infinitesimal transformations

$$Y_1 f = \frac{\partial f}{\partial y_2} \ , \ Y_2 f = \frac{\partial f}{\partial y_3} \ , \ Y_3 f = -\frac{\partial f}{\partial y_1} + y_2 \frac{\partial f}{\partial y_2} + hy_3 \frac{\partial f}{\partial y_3}$$

and if, along with the hypothesized equations of correspondence between the two spaces, we assume that[55] $X_1 f, X_2 f, X_3 f$ are changed respectively into $\bar{Y}_1 f, \bar{Y}_2 f, \bar{Y}_3 f$, then these latter ones must be combinations of $Y_1 f, Y_2 f, Y_3 f$ and have the same composition

$$[\bar{Y}_1, \bar{Y}_2]f = 0 \ , \ [\bar{Y}_1, \bar{Y}_3]f = \bar{Y}_1 f \ , \ [\bar{Y}_2, \bar{Y}_3]f = h\bar{Y}_2 f \ .$$

[54] Bianchi's obvious typo was corrected here, there was a "+" between e^{2x_1} and dx_2^2 [Editor].

[55] The original paper has $Y_1 f$, $Y_2 f$ and $Y_3 f$ here; correction based on the *Opere* [Editor].

It is clear in the first place that $\bar{Y}_1 f, \bar{Y}_2 f$ must be combinations of $Y_1 f, Y_2 f$ only, so that we will have

$$\bar{Y}_1 f = \alpha Y_1 f + \beta Y_2 f \ , \ \bar{Y}_2 f = \gamma Y_1 f + \delta Y_2 f \ ,$$
$$\bar{Y}_3 f = a Y_1 f + b Y_2 f + c Y_3 f \ .$$

Taking into account the composition equations we see that for $h \neq -1$ one necessarily has $\beta = \gamma = 0$, $c = 1$, so that it follows that

$$\bar{Y}_1 f = \alpha Y_1 f \ , \ \bar{Y}_2 f = \delta Y_2 f \ , \ \bar{Y}_3 f = a Y_1 f + b Y_2 f + Y_3 f \ .$$

while for $h = -1$ there is also possible another case

$$\bar{Y}_1 f = \beta Y_2 f \ , \ \bar{Y}_2 f = \gamma Y_1 f \ , \ \bar{Y}_3 f = a Y_1 f + b Y_2 f - Y_3 f \ ,$$

which does not differ from the previous one, however, apart from the exchange of y_2 and y_3 and the change of y_1 into $-y_1$ (this clearly does not change the line element). We can therefore limit ourselves to the first case, in which by integrating the equations of the transformations we find

$$y_1 = x_1 + k \ , \ y_2 = \alpha x_2 + l e^{-x_1} - a \ , \ y_3 = \delta x_3 + p e^{-h x_1} - b/h \ ,$$

where k, l, p indicate new constants. Substituting into the line element (β) we obtain

$$\begin{aligned} & dx_1^2 + e^{2x_1+2k} (\alpha\, dx_2 - l e^{-x_1}\, dx_1)^2 \\ & + 2m e^{(h+1)(x_1+k)} (\alpha\, dx_2 - l e^{-x_1}\, dx_1)(\delta\, dx_3 - h p e^{-h x_1}\, dx_1) \\ & + 2 e^{2h(x_1+k)} (\delta\, dx_3 - h p e^{-h x_1}\, dx_1)^2 \ . \end{aligned}$$

Expressing the fact that this differential form differs from (α) only by a constant factor, it suffices to compare the coefficients of dx_2^2, $dx_2 dx_3$, dx_3^2 to find $\alpha^2 e^{2k} = \delta^2 e^{2hk} = (m/n) e^{(h+1)k} \alpha\delta$, from which it indeed follows that $n^2 = m^2$, Q.E.D.

25 The groups of type VII$_1$: $[X_1, X_2]f = 0,\ [X_1, X_3]f = X_2 f,\ [X_2, X_3]f = -X_1 f$.

Treating in general the case of the groups of type VII of composition

$$[X_1, X_2]f = 0,\ [X_1, X_3]f = X_2 f,\ [X_2, X_3]f = -X_1 f + h X_2 f \ ,$$

we must give to the constants a, b, c, a', b', c' of §14 the values $a = 0$, $b = 1$, $c = 0$, $a' = -1$, $b' = h$, $c' = 0$, from which one has $\xi_1 =$ *constant*, so we set $\xi_1 = 1/k$ and equations (F) (*ibid.*) give us

$$\alpha' + 2k\beta = 0 \ , \ \gamma' - 2k\beta + 2hk\gamma = 0 \ , \ \beta' - k\alpha + hk\beta + k\gamma = 0 \ . \tag{83}$$

For the integration it is convenient to separate the case $h = 0$ from the general case. We assume $h = 0$ in this section and integrating (83) we will have

$$\alpha = c_1 \sin(2kx_1) + c_2 \cos(2kx_1) + c_3 \ ,$$
$$\beta = -c_1 \cos(2kx_1) + c_2 \sin(2kx_1) \ ,$$
$$\gamma = -c_1 \sin(2kx_1) - c_2 \cos(2kx_1) + c_3 \ ,$$

where c_1, c_2, c_3 are three constants. We exclude the case in which the first two are both zero since then the space would be of zero curvature. Changing x_1 into $x_1 + \textit{constant}$, we can make $c_1 = 0$ and varying the parameters x_2, x_3 proportionally we can make $c_2 = 1$; finally by replacing the space with a similar space, we will have the following standard form for the line element:

$$ds^2 = dx_1^2 + (n + \cos x_1)\, dx_2^2 + 2 \sin x_1\ dx_2 dx_3 + (n - \cos x_1)\, dx_3^2 \ , \tag{84}$$

where the constant n will be positive and > 1 since α, γ, $\alpha\gamma - \beta^2$ must be positive. The equations (E) §14 to determine η_1, η_2, η_3 become:

$$\frac{\partial \eta_1}{\partial x_1} = 0 \ , \tag{85}$$

$$\frac{\partial \eta_1}{\partial x_2} + (n + \cos x_1)\frac{\partial \eta_2}{\partial x_1} + \sin x_1 \frac{\partial \eta_3}{\partial x_1} = 0 \ , \tag{86a}$$

$$\frac{\partial \eta_1}{\partial x_3} + \sin x_1 \frac{\partial \eta_2}{\partial x_1} + (n - \cos x_1)\frac{\partial \eta_3}{\partial x_1} = 0 \ , \tag{86b}$$

$$-\frac{1}{2} \sin x_1 \cdot \eta_1 + (n + \cos x_1)\frac{\partial \eta_2}{\partial x_2} + \sin x_1 \frac{\partial \eta_3}{\partial x_2} = 0 \ , \tag{87}$$

$$\frac{1}{2} \sin x_1 \cdot \eta_1 + \sin x_1 \frac{\partial \eta_2}{\partial x_3} + (n - \cos x_1)\frac{\partial \eta_3}{\partial x_3} = 0 \ , \tag{88}$$

$$\begin{aligned} &\cos x_1 \eta_1 + (n + \cos x_1)\frac{\partial \eta_2}{\partial x_3} + \sin x_1 \left(\frac{\partial \eta_2}{\partial x_2} + \frac{\partial \eta_3}{\partial x_3}\right) \\ &\qquad + (n - \cos x_1)\frac{\partial \eta_3}{\partial x_2} = 0 \ . \end{aligned} \tag{89}$$

Solving (86) for $\partial\eta_2/\partial x_1$, $\partial\eta_3/\partial x_1$ and integrating we have

$$\begin{aligned} \eta_2 &= \frac{1}{n^2 - 1}\left\{(\sin x_1 - n x_1)\frac{\partial \eta_1}{\partial x_2} - \cos x_1 \frac{\partial \eta_1}{\partial x_3}\right\} + \psi(x_2, x_3) \ , \\ \eta_3 &= \frac{1}{n^2 - 1}\left\{-(\sin x_1 + n x_1)\frac{\partial \eta_1}{\partial x_3} - \cos x_1 \frac{\partial \eta_1}{\partial x_2}\right\} + \chi(x_2, x_3) \ , \end{aligned}$$

and substituting these values into the successive equations we see that one must have

$$\begin{aligned} &\frac{\partial^2 \eta_1}{\partial {x_2}^2} = \frac{\partial^2 \eta_1}{\partial x_2 \partial x_3} = \frac{\partial^2 \eta_1}{\partial {x_3}^2} = 0 \ , \\ &\frac{\partial \psi}{\partial x_2} = 0 \ , \ \frac{\partial \psi}{\partial x_3} = -\frac{1}{2}\eta_1 \ , \ \frac{\partial \chi}{\partial x_2} = \frac{1}{2}\eta_1 \ , \ \frac{\partial \chi}{\partial x_3} = 0 \ , \end{aligned}$$

so that we obtain:

$$\eta_1 = a \ , \ \eta_2 = -\frac{a}{2} x_3 + b \ , \ \eta_3 = \frac{a}{2} x_2 + c \ ,$$

with a, b, c constants. Here too the group of motions is of three parameters and its generating transformations are

$$X_1 f = \frac{\partial f}{\partial x_2} \ , \ X_2 f = \frac{\partial f}{\partial x_3} \ , \ X_3 f = 2\frac{\partial f}{\partial x_1} - x_3 \frac{\partial f}{\partial x_2} + x_2 \frac{\partial f}{\partial x_3} \ , \tag{90}$$

with the composition

$$[X_1, X_2] f = 0 \ , \ [X_1, X_3] f = X_2 f \ , \ [X_2, X_3] f = -X_1 f \ .$$

26 The groups of type VII_2:

$[X_1, X_2]f = 0$, $[X_1, X_3]f = X_2 f$, $[X_2, X_3]f = -X_1 f + hX_2 f$, $h \neq 0$ $(0 < h < 2)$.

The equations (83)[56] give us

$$\beta = -\frac{\alpha'}{2k} \ , \ \gamma = \alpha + \frac{\alpha''}{2k^2} + \frac{h\alpha'}{2k} \ ,$$

and hence to determine α, the linear and homogeneous constant coefficient equation

$$\alpha''' + 3hk\alpha'' + 2k^2(h^2+2)\alpha' + 4hk^3\alpha = 0 \ ,$$

whose characteristic equation

$$\rho^3 + 3hk\rho^2 + 2k^2(h^2+2)\rho + 4hk^3 = 0 \ ,$$

setting $\rho = kr$, becomes

$$r^3 + 3hr^2 + 2(h^2+2)r + 4h = 0 \ .$$

.

One root of this equation is $r_1 = -h$ and the other two r_2, r_3, since $h^2 < 4$, are complex conjugates:

$$r_2 = -h + i\sqrt{4-h^2} \ , \ r_3 = -h - i\sqrt{4-h^2} \ .$$

If for brevity we set $v = \sqrt{4-h^2}$, we have for α the expression:

$$\alpha = c_1 e^{-hkx_1} + c_2 e^{-hkx_1}\cos(kvx_1) + c_3 e^{-hkx_1}\sin(kvx_1) \ ,$$

where c_1, c_2, c_3 are three arbitrary constants.

We exclude the case in which one takes $c_2 = c_3 = 0$ because then the space would be of constant negative curvature. By adding a constant to x_1 we can make (if $c_2 \neq 0$) $c_3 = 0$, and passing to a similar space we will obtain

$$\begin{aligned}
\alpha &= e^{-hx_1}(n + \cos vx_1) \ , \\
\beta &= \frac{1}{2}e^{-hx_1}(h\cos vx_1 + v\sin vx_1 + nh) \ , \\
\gamma &= e^{-hkx_1}\left(\frac{2-v^2}{2}\cos vx_1 + \frac{hv}{2}\sin vx_1 + n\right) \ .
\end{aligned}$$

We note that from this follows the result

$$\alpha\gamma - \beta^2 = v(n^2-1)e^{-2hx_1}/4 \ ,$$

from which $|n| > 1$ so that $n > 0$ since $\alpha > 0$.

The first equations of (E) §14, solved for $\partial\eta_2/\partial x_1$, $\partial\eta_3/\partial x_1$, give

$$\begin{aligned}
\frac{\partial\eta_2}{\partial x_1} &= \frac{4e^{2hx_1}}{(n^2-1)v^2}\left\{\beta\frac{\partial\eta_1}{\partial x_3} - \gamma\frac{\partial\eta_1}{\partial x_2}\right\} \ , \\
\frac{\partial\eta_3}{\partial x_1} &= \frac{4e^{2hx_1}}{(n^2-1)v^2}\left\{\beta\frac{\partial\eta_1}{\partial x_2} - \alpha\frac{\partial\eta_1}{\partial x_3}\right\} \ .
\end{aligned}$$

[56] In the original paper, "equations (84)", which is incorrect [Editor].

Substituting the values of α, β, γ and integrating with respect to x_1 we obtain

$$\eta_2 = \left\{ \frac{h^2 - v^2}{2v^2(n^2-1)} \cos vx_1 + \frac{h}{(n^2-1)v} \sin vx_1 + \frac{2n}{v^2(n^2-1)} \right\} e^{hx_1} \frac{\partial \eta_1}{\partial x_3}$$
$$- \left\{ \frac{h(1-v^2)}{v^2(n^2-1)} \cos vx_1 + \frac{h^2-1}{(n^2-1)v} \sin vx_1 + \frac{4n}{v^2 h(n^2-1)} \right\} e^{hx_1} \frac{\partial \eta_1}{\partial x_2}$$
$$+\psi(x_2, x_3) \ ,$$
$$\eta_3 = \left\{ \frac{h^2 - v^2}{2v^2(n^2-1)} \cos vx_1 + \frac{h}{(n^2-1)v} \sin vx_1 + \frac{2n}{v^2(n^2-1)} \right\} e^{hx_1} \frac{\partial \eta_1}{\partial x_2}$$
$$- \left\{ \frac{h}{v^2(n^2-1)} \cos vx_1 + \frac{1}{(n^2-1)v} \sin vx_1 + \frac{4n}{v^2 h(n^2-1)} \right\} e^{hx_1} \frac{\partial \eta_1}{\partial x_3}$$
$$+\chi(x_2, x_3) \ .$$

If we now take the other three equations (E) §14:

$$\gamma'\eta_1 + 2\beta \frac{\partial \eta_2}{\partial x_3} + 2\gamma \frac{\partial \eta_3}{\partial x_3} = 0 \ , \ \alpha'\eta_1 + 2\alpha \frac{\partial \eta_2}{\partial x_2} + 2\beta \frac{\partial \eta_3}{\partial x_2} = 0 \ ,$$
$$\beta'\eta_1 + 2\alpha \frac{\partial \eta_2}{\partial x_3} + \beta \left(\frac{\partial \eta_2}{\partial x_2} + \frac{\partial \eta_3}{\partial x_3} \right) + \gamma \frac{\partial \eta_3}{\partial x_2} = 0$$

and substitute the values of $\alpha, \beta, \gamma, \eta_2, \eta_3$ into them, it suffices to equate the coefficients of the terms in e^{-hx_1}, $e^{-hx_1} \cos vx_1$, $e^{-hx_1} \sin vx_1$, to find

$$\frac{\partial \psi}{\partial x_2} = 0 \ , \ \frac{\partial \psi}{\partial x_3} = -\eta_1 \ , \ \frac{\partial \chi}{\partial x_2} = \eta_1 \ , \ \frac{\partial \chi}{\partial x_3} = h\eta_1 \ ,$$

from which it follows that

$$\eta_1 = a \ , \ \eta_2 = -ax_3 + b \ , \ \eta_3 = ax_2 + ahx_3 + c \ ,$$

with a, b, c arbitrary constants. Therefore in the present case the space has as a group of motions the G_3 generated by the three infinitesimal transformations:

$$X_1 f = \frac{\partial f}{\partial x_2} \ , \ X_2 f = \frac{\partial f}{\partial x_3} \ , \ X_3 f = \frac{\partial f}{\partial x_1} - x_3 \frac{\partial f}{\partial x_2} + (x_2 + hx_3) \frac{\partial f}{\partial x_3} \ ,$$

with the composition

$$[X_1, X_2]f = 0 \ , \ [X_1, X_3]f = X_2 f \ , \ [X_2, X_3]f = -X_1 f + hX_2 f \ .$$

We see that setting $h = 0$ one returns to the results of the previous section, changing the notation in a very simple way.

27 The constant n is essential in the line elements of the two previous sections.

In the line element of the spaces of the previous section appear the two constants h, n, the first of which is essential, already being so by the composition of the group (§13). We now show that the constant n is essential, and with this result the same thing will also be proved for the spaces of §25 which correspond to $h = 0$.

We must show that two spaces of respective line elements:

$$
\begin{aligned}
ds^2 &= dx_1^2 + e^{-hx_1}(n+\cos vx_1)\,dx_2^2 \\
&\quad + e^{-hx_1}(h\cos vx_1 + v\sin vx_1 + nh)\,dx_2dx_3 \\
&\quad + e^{-hx_1}\left(\frac{2-v^2}{2}\cos vx_1 + \frac{hv}{2}\sin vx_1 + n\right)\,dx_3^2 \ , \\
ds^2 &= dy_1^2 + e^{-hy_1}(m+\cos vy_1)\,dy_2^2 \\
&\quad + e^{-hy_1}(h\cos vy_1 + v\sin vy_1 + mh)\,dy_2dy_3 \\
&\quad + e^{-hy_1}\left(\frac{2-v^2}{2}\cos vy_1 + \frac{hv}{2}\sin vy_1 + m\right)\,dy_3^2 \ ,
\end{aligned}
$$

cannot be similar unless $n^2 = m^2$.

The group G_3 of motions of the first space is generated by the infinitesimal transformations (90) and the Γ_3 of the second by the three

$$
Y_1f = \frac{\partial f}{\partial y_2}\ , Y_2f = \frac{\partial f}{\partial y_3}\ , Y_3f = \frac{\partial f}{\partial y_1} - y_3\frac{\partial f}{\partial y_2} + (y_2 + hy_3)\frac{\partial f}{\partial y_3}\ ,
$$

with the same composition. Suppose that in the hypothesized transformation X_1f, X_2f, X_3f are changed into $\bar{Y}_1f, \bar{Y}_2f, \bar{Y}_3f$; we will have:

$$
\begin{aligned}
&\bar{Y}_1f = \alpha Y_1f + \beta Y_2f\ , \bar{Y}_2f = \gamma Y_1f + \delta Y_2f\ , \\
&\bar{Y}_3f = aY_1f + bY_2f + cY_3f\ .
\end{aligned}
$$

From the composition equations

$$
[\bar{Y}_1, \bar{Y}_2]f = 0\ ,\ [\bar{Y}_1, \bar{Y}_3]f = \bar{Y}_2f\ ,\ [\bar{Y}_2, \bar{Y}_3]f = -\bar{Y}_1f + h\bar{Y}_2f\ ,
$$

it immediately follows that $c = 1$, $\gamma = -\beta$, $\delta = \alpha + h\beta$, so

$$
\begin{aligned}
&\bar{Y}_1f = \alpha Y_1f + \beta Y_2f\ ,\ \bar{Y}_2f = -\beta Y_1f + (\alpha + h\beta)Y_2f\ , \\
&\bar{Y}_3f = aY_1f + bY_2f + Y_3f\ .
\end{aligned}
$$

When the y are expressed in terms of the x, they must consequently satisfy the following equations:

$$
\begin{aligned}
\frac{\partial y_1}{\partial x_1} = 1\ , \qquad & \frac{\partial y_1}{\partial x_2} = 0\ ,\ \frac{\partial y_1}{\partial x_3} = 0\ , \\
& \frac{\partial y_2}{\partial x_2} = \alpha\ ,\ \frac{\partial y_2}{\partial x_3} = -\beta\ , \\
& \frac{\partial y_3}{\partial x_2} = \beta\ ,\ \frac{\partial y_3}{\partial x_3} = \alpha + h\beta\ .
\end{aligned}
$$

It suffices to compare the terms in dx_2^2 in the two line elements (91), (92) to obtain the following equation, in which λ denotes a constant factor:

$$
\begin{aligned}
&\alpha^2(\cos vy_1 + m) + \alpha\beta(h\cos vy_1 + v\sin vx_1 + hm) \\
&\qquad + \beta^2\left(\frac{2-v^2}{2}\cos vy_1 + \frac{hv}{2}\sin vy_1 + m\right) \\
&= \lambda(\cos vx_1 + n)\ .
\end{aligned}
$$

This must be converted into an identity in x_1 by setting $y_1 = x_1 + k$ (k constant). Setting $vk = \sigma$ (constant), and comparing corresponding terms in the above equations, we derive the three relations

$$\alpha^2 + h\alpha\beta + \beta^2 = \lambda n/m \tag{93}$$

$$\begin{aligned}
&\alpha^2 \cos\sigma + \alpha\beta(h\cos\sigma + v\sin\sigma)\\
&\qquad +\beta^2\left(\frac{2-v^2}{2}\cos\sigma + \frac{hv}{2}\sin\sigma\right) = \lambda\ ,\\
&-\alpha^2 \sin\sigma + \alpha\beta(-h\sin\sigma + v\cos\sigma)\\
&\qquad +\beta^2\left(-\frac{2-v^2}{2}\sin\sigma + \frac{hv}{2}\cos\sigma\right) = 0\ .
\end{aligned}$$

Multiplying respectively the last two equations, first by $\cos\sigma$, $-\sin\sigma$ then by $\sin\sigma$, $\cos\sigma$, and each time summing, we obtain

$$\alpha^2 + h\alpha\beta + \frac{2-v^2}{2}\beta^2 = \lambda\cos\sigma\ ,\ v\alpha\beta + \frac{hv}{2}\beta^2 = \lambda\sin\sigma\ ,$$

which squared and summed, remembering that $v^2 + h^2 = 4$ give $(\alpha^2 + h\alpha\beta + \beta^2)^2 = \lambda^2$, from which by (93) $n^2 = m^2$, Q.E.D.

28 The groups of type VIII: $[X_1, X_2]f = X_1 f,\ [X_1, X_3]f = 2X_2 f,\ [X_2, X_3]f = X_3 f$.

Having exhausted the research on spaces which admit an integrable transitive G_3 of motions, we now turn to the case of a simple transitive G_3, beginning with type VIII.

We consider in G_3 the G_2 generated by $X_2 f, X_3 f$ and proceed as in §4 by assuming the geodesically parallel surfaces invariant with respect to the subgroup G_2 as the coordinate surfaces $x_1 = constant$, and we furthermore give to $X_2 f, X_3 f$ the canonical form (*ibid.*)

$$X_2 f = \partial f/\partial x_3\ ,\ X_3 f = e^{x_3}\partial f/\partial x_2\ .$$

For the line element of the space we therefore have

$$ds^2 = dx_1^2 + \alpha\, dx_2^2 + 2(\beta - \alpha x_2)\, dx_2 dx_3 + (\alpha x_2^2 - 2\beta x_2 + \gamma)\, dx_3^2\ , \tag{94}$$

with α, β, γ functions of x_1.

Now let $X_1 f = \xi_1\,\partial f/\partial x_1 + \xi_2\,\partial f/\partial x_2 + \xi_3\,\partial f/\partial x_3$ be the third generating transformation of G_3, in which, the group being transitive, we will have $\xi_1 \neq 0$. Because the composition equations $[X_1, X_2]f = X_1 f$, $[X_1, X_3]f = 2X_2 f$ hold, the ξ must satisfy the following equations:

$$\begin{aligned}
&\frac{\partial\xi_1}{\partial x_3} = -\xi_1\ ,\ \frac{\partial\xi_2}{\partial x_3} = -\xi_2\ ,\ \frac{\partial\xi_3}{\partial x_3} = -\xi_3\ ,\\
&\frac{\partial\xi_1}{\partial x_2} = 0\ ,\ \frac{\partial\xi_2}{\partial x_2} = \xi_3\ ,\ \frac{\partial\xi_3}{\partial x_2} = -2e^{-x_3}\ ,
\end{aligned}$$

from which integrating leads to

$$\xi_1 = Ae^{-x_3}\ ,\ \xi_2 = (Bx_2 - x_2^2 + C)e^{-x_3}\ ,\ \xi_3 = (B - 2x_2)e^{-x_3}\ , \tag{95}$$

with A, B, C functions only of x_1.

Expressing the fact that, with the values (95) of the ξ and assuming

$$a_{11} = 1 \ , \ a_{12} = a_{13} = 0 \ ,$$
$$a_{22} = \alpha \ , a_{23} = \beta - \alpha x_2 \ , \ a_{33} = \alpha x_2^2 - 2\beta x_2 + \gamma \ ,$$

the fundamental equations (A) §1 are satisfied, we find among the unknown functions $\alpha, \beta, \gamma, A, B, C$ of x_1 the 6 following equations:

$$A' = 0 \ ,$$
$$\alpha C' + \beta B' = 0 \ , \ \beta C' + \gamma B' = A \ ,$$
$$\frac{1}{2} A\alpha' + \alpha B - 2\beta = 0 \ , \ \frac{1}{2} A\beta' - \alpha C - \gamma = 0 \ , \ \frac{1}{2} A\gamma' - 2\beta C - \gamma B = 0 \ .$$

The first tells us that A is a constant, different from zero by hypothesis; then multiplying the last three respectively by γ, -2β, α and summing leads to $A(\alpha'\gamma + \alpha\gamma' - 2\beta\beta') = 0$ so that $\alpha\gamma - \beta^2 = constant$.

We therefore set

$$A = 2k \ , \ \alpha\gamma - \beta^2 = n^2 \tag{96}$$

and it follows that α, β, γ are expressed in terms of B', C' by the formulas

$$\alpha = \frac{n^2}{2k} B' \ , \ \beta = -\frac{n^2}{2k} C' \ , \ \gamma = \frac{2k}{B'} + \frac{n^2}{2k} \frac{C'^2}{B'} \ , \tag{97}$$

while B, C must satisfy the simultaneous second order differential equations:

$$kB'' + BB' + 2C' = 0 \ , \ n^2 B' C'' + \frac{n^2}{k} B'^2 C + \frac{n^2}{k} C'^2 + 4k = 0 \ .$$

The first of these is immediately integrable, and indicating by $2a$ the constant of integration, we find

$$C = a - \frac{k}{2} B' - \frac{1}{4} B^2 \ . \tag{98}$$

Finally by substituting this into the last one we have, to determine $B(x_1)$, the third order differential equation

$$-\frac{kn^2}{4} B' B''' + \frac{kn^2}{8} B''^2 - \frac{n^2}{2} B'^3 + \frac{an^2}{2k} B'^2 + 2k = 0 \ .$$

Having integrated this, (98) gives us the value of C and (97) those of α, β, γ in the line element (94) of the space.

We treat in this section the particular case in which B' is constant, namely $B'' = 0$, a case which returns us to the spaces already considered in §17. We will have

$$B' = l \ , \ B = lx_1 + m \ , \ C' = -l(lx_1 + m)/2 \ ,$$

with l, m constants,[57] so by (97):

$$\alpha = \frac{n^2 l}{2k} \ , \ \beta = \frac{n^2 l}{2k} \frac{lx_1 + m}{2} \ , \ \gamma = \frac{2k}{l} + \frac{n^2 l}{2k} \left(\frac{lx_1 + m}{2} \right)^2 \ .$$

[57]One observes that the constant l cannot be zero because then we would have $B' = C' = 0$, and consequently $A = 0$.

Setting $(lx_1+m)/2=y_1$, the line element (94) becomes

$$ds^2 \quad = \quad \frac{4}{l^2}\,dy_1^2+\frac{n^2 l}{2k}\left\{dx_2^2+2(y_1-x_2)\,dx_2dx_3+\left[(y_1-x_2)^2+\frac{4k^2}{n^2l^2}\right]\,dx_3^2\right\}\ ,$$

and passing to a similar space by dividing by $n^2l/2k$:

$$ds^2=a^2\,dy_1^2+dx_2^2+2(y_1-x_2)\,dx_2dx_3+\left\{(y_1-x_2)^2+b^2\right\}\,dx_3^2\ ,$$

a,b being constants. We now set $y_1=b/a\,z_1$, $x_2=by_2$ and dividing by b^2 leads to

$$ds^2=dz_1^2+dy_2^2+2\left(\frac{z_1}{a}-y_2\right)\,dy_2dx_3+\left\{\left(\frac{z_1}{a}-y_2\right)^2+1\right\}\,dx_3^2\ ,$$

a formula which differs only in notation from (49*) of §17. Therefore, in the case $B''=0$, the group of motions of the space is a G_4 of composition already examined.

29 Integration in the general case by elliptic functions.

We now treat the general case in which B' is variable, therefore B'' as well because of the differential equation (99).

We immediately reduce this equation to a quadrature, assuming as the independent variable $B'=s$ and taking $B''^2=t$ for the unknown function. In this way (99) becomes

$$-s\frac{dt}{ds}+t=\frac{4s^3}{k}-\frac{4as^2}{k^2}-\frac{16}{n^2}\ ,$$

from which by integrating

$$t=-\frac{2s^3}{k}+\frac{4as^2}{k^2}+cs-\frac{16}{n^2}\ ,$$

with c a new arbitrary constant. We have therefore

$$B''=\frac{ds}{dx_1}=\sqrt{-\frac{2s^3}{k}+\frac{4as^2}{k^2}+cs-\frac{16}{n^2}}\ ,$$

namely

$$x_1=\int\frac{ds}{\sqrt{-\frac{2s^2}{k}+\frac{4as^2}{k^2}+cs-\frac{16}{n^2}}}\ .$$

We integrate this by introducing the Weierstrass elliptical function[58] $\mathcal{P}(x_1)$ with the invariants

$$g_2=\frac{4a^2}{3k^4}+\frac{c}{2k}\ ,\ g_3=\frac{4}{n^2k^2}-\frac{8a^3}{27k^6}-\frac{ac}{6k^3}\ , \tag{100}$$

and neglecting the additive constant in x_1 as is permissible, we will have

$$B'=s=\frac{2a}{3k}-2k\mathcal{P}(x_1)\ . \tag{101}$$

[58]Bianchi's notation $\mathcal{P}x_1$, ζx_1 was changed to the now-common $\mathcal{P}(x_1)$, $\zeta(x_1)$ [Editor].

Integrating again we introduce the Weierstrass function $\zeta(x_1) = \frac{\sigma'(x_1)}{\sigma(x_1)}$ and one has

$$B = \frac{2a}{3k}x_1 - 2k\zeta(x_1) + h \ , \tag{102}$$

with h a new constant, so that from (98) we have

$$C = \frac{2a}{3} + k^2\mathcal{P}(x_1) - \frac{1}{4}\left\{\frac{2a}{3k}x_1 + 2k\zeta(x_1) + h\right\}^2 \ , \tag{103a}$$

$$C' = k^2\mathcal{P}'(x_1) - \left(\frac{a}{3k} - k\mathcal{P}(x_1)\right)\left\{\frac{2a}{3k}x_1 + 2k\zeta(x_1) + h\right\} \ . \tag{103b}$$

Equations (97) then give us immediately for the values of α, β which appear in the line element

$$\alpha = \frac{an^2}{3k^2} - n^2\mathcal{P}(x_1) \ , \tag{104a}$$

$$\beta = -\frac{kn^2}{2}\mathcal{P}'(x_1) + \frac{n^2}{2k}\left(\frac{a}{3k} - k\mathcal{P}(x_1)\right)\left(\frac{2a}{3k}x_1 + 2k\zeta(x_1) + h\right) \ . \tag{104b}$$

The value of γ appears above instead in fractional form with the denominator $B' = 2k(a/(3k^2) - \mathcal{P}(x_1))$, but if we transform it, taking into account the relation $\mathcal{P}'^2(x_1) = 4\mathcal{P}^3(x_1) - g_2\mathcal{P}(x_1) - g_3$ and applying it to the values (100) of the invariants we find

$$\begin{aligned}\gamma = &-k^2n^2\mathcal{P}^2(x_1) - \frac{an^2}{3}\mathcal{P}(x_1) + \frac{2a^2n^2}{9k^2} + \frac{ck^2n^2}{8} \\ &+\frac{n^2}{4k}\left(\frac{a}{3k} - k\mathcal{P}(x_1)\right)\left(\frac{2a}{3k}x_1 + 2k\zeta(x_1) + h\right)^2 \\ &-\frac{kn^2}{2}\mathcal{P}'(x_1)\left(\frac{2a}{3k}x_1 + 2k\zeta(x_1) + h\right) \ .\end{aligned} \tag{105}$$

It is worth noting that, in view of the relation $\mathcal{P}''(x_1) = 6\mathcal{P}^2(x_1) - g_2/2$, the derivative of β has the following value:[59]

$$\begin{aligned}\beta' = &\frac{cn^2}{8} + \frac{4a^2n^2}{9k^3} - \frac{2an^2}{3k}\mathcal{P}(x_1) - 2kn^2\mathcal{P}^2(x_1) \\ &-\frac{n^2}{2}\mathcal{P}'(x_1)\left(\frac{2a}{3k}x_1 + 2k\zeta(x_1) + h\right) \ .\end{aligned} \tag{106}$$

[59] The preceding formulas can be greatly simplified by observing that without loss of generality one can set $h = 0$, $n = k = 1$, as follows from simple considerations. Then $e = a/3$ is a root of the equation $4e^3 - g_2e - g_3 = 0$ and one has

$$\alpha = e - \mathcal{P}(x_1) \ , \ \beta = -\frac{1}{2}\mathcal{P}(x_1) + (e - \mathcal{P}(x_1))(ex_1 + \zeta(x_1)) \ ,$$

$$\gamma = -\mathcal{P}^2(x_1) - e\mathcal{P}(x_1) + e^2 + \frac{g_2}{4} + (e - \mathcal{P}(x_1))(ex_1 + \zeta(x_1))^2 \ .$$

30 The most general group of motions of the space of the previous section.

To find the most general infinitesimal motion $Xf = \xi_1\,\partial f/\partial x_1 + \xi_2\,\partial f/\partial x_2 + \xi_3\,\partial f/\partial x_3$ of our space we recall the fundamental equations (A) §1. First setting $i = k = 1$ we have $\partial\xi_1/\partial x_1 = 0$, which shows that ξ_1 does not depend on x_1. The remaining equations give us

$$\frac{\partial\xi_1}{\partial x_2} + \alpha\frac{\partial\xi_2}{\partial x_1} + (\beta - \alpha x_2)\frac{\partial\xi_3}{\partial x_1} = 0\ , \tag{107a}$$

$$\frac{\partial\xi_1}{\partial x_3} + (\beta - \alpha x_2)\frac{\partial\xi_2}{\partial x_1} + (\alpha x_2^2 - 2\beta x_2 + \gamma)\frac{\partial\xi_3}{\partial x_1} = 0\ , \tag{107b}$$

$$\frac{1}{2}\alpha'\xi_1 + \alpha\frac{\partial\xi_2}{\partial x_2} + (\beta - \alpha x_2)\frac{\partial\xi_3}{\partial x_2} = 0\ , \tag{108}$$

$$\frac{1}{2}(\alpha' x_2^2 - 2\beta' x_2 + \gamma')\xi_1 - (\beta - \alpha x_2)\xi_2 + (\beta - \alpha x_2)\frac{\partial\xi_2}{\partial x_3} + (\alpha x_2^2 - 2\beta x_2 + \gamma)\frac{\partial\xi_3}{\partial x_3} = 0\ , \tag{109}$$

$$(\beta' - \alpha' x_2)\xi_1 - \alpha\xi_2 + \alpha\frac{\partial\xi_2}{\partial x_3} + (\beta - \alpha x_2)\left(\frac{\partial\xi_2}{\partial x_2} + \frac{\partial\xi_3}{\partial x_3}\right) + (\alpha x_2^2 - 2\beta x_2 + \gamma)\frac{\partial\xi_3}{\partial x_2} = 0\ . \tag{110}$$

Solving (107) for $\partial\xi_2/\partial x_1$, $\partial\xi_3/\partial x_1$ we have:

$$\frac{\partial\xi_2}{\partial x_1} = \frac{1}{n_2}\left\{(\beta - \alpha x_2)\frac{\partial\xi_1}{\partial x_3} - (\alpha x_2^2 - 2\beta x_2 + \gamma)\frac{\partial\xi_1}{\partial x_2}\right\}\ , \tag{111a}$$

$$\frac{\partial\xi_3}{\partial x_1} = \frac{1}{n_2}\left\{(\beta - \alpha x_2)\frac{\partial\xi_1}{\partial x_2} - \alpha\frac{\partial\xi_1}{\partial x_3}\right\}\ . \tag{111b}$$

We integrate the preceding equations with respect to x_1, and for brevity set $\alpha_0 = \int\alpha\,dx_1$, $\beta_0 = \int\beta\,dx_1$, $\gamma_0 = \int\gamma\,dx_1$, fixing, however, the additive constants in α_0, β_0, γ_0: we assume, according to (104)

$$\alpha_0 = \frac{an^2}{3k^2}x_1 + n^2\zeta(x_1)\ , \tag{112a}$$

$$\beta_0 = -\frac{kn^2}{2}\mathcal{P}(x_1) + \frac{n^2}{8k}\left(\frac{2a}{3k}x_1 + 2k\zeta(x_1) + h\right)^2\ . \tag{112b}$$

Regarding the value of γ_0, we need only observe that by formula (105) it contains terms that cannot in any way be eliminated with those arising from α_0, β_0. Given this, integrating (111) we have

$$\xi_2 = \frac{1}{n^2}\left\{(\beta_0 - \alpha_0 x_2)\frac{\partial\xi_1}{\partial x_3} - (\alpha_0 x_2^2 - 2\beta_0 x_2 + \gamma_0)\frac{\partial\xi_1}{\partial x_2}\right\} + \psi(x_2, x_3)\ , \tag{113}$$

$$\xi_3 = \frac{1}{n^2}\left\{(\beta_0 - \alpha_0 x_2)\frac{\partial\xi_1}{\partial x_2} - \alpha_0\frac{\partial\xi_1}{\partial x_3}\right\} + \chi(x_2, x_3)\ .$$

Substituting into (108), (109), (110) all those terms which contain γ_0 must be zero separately by the observation made above; from this we then obtain $\partial^2\xi_1/\partial {x_2}^2 = 0$,

$\partial^2\xi_1/\partial x_2\partial x_3 = \partial\xi_1/\partial x_2$, so that (108) becomes:

$$\frac{1}{2}\alpha'\xi_1 + \frac{3\alpha}{n^2}(\beta_0 - \alpha_0 x_2)\frac{\partial\xi_1}{\partial x_2} - \frac{2\alpha_0}{n^2}(\beta - \alpha x_2)\frac{\partial\xi_1}{\partial x_2} - \frac{\alpha\alpha_0}{n^2}\frac{\partial\xi_1}{\partial x_3}$$
$$+(\beta - \alpha x_2)\frac{\partial\chi}{\partial x_2} + \alpha\frac{\partial\psi}{\partial x_2} = 0 \ .$$

If we observe that in this the term in $\zeta^2(x_1)$, arising from β_0, cannot be cancelled by any other, we see that we must have

$$\frac{\partial\xi_1}{\partial x_2} = 0 \ , \tag{114}$$

after which the previous equation becomes

$$-\frac{n^2}{2}\mathcal{P}'(x_1)\xi_1 + \left\{\frac{n^2}{2k}\left(\frac{a}{3k} - k\mathcal{P}(x_1)\right)\left(\frac{2a}{3k}x_1 + 2k\zeta(x_1) + h\right) - \frac{kn^2}{2}\mathcal{P}'(x_1)\right\}\frac{\partial\chi}{\partial x_2}$$
$$-\frac{n^2}{k}\left(\frac{a}{3k} - k\mathcal{P}(x_1)\right)x_2\frac{\partial\chi}{\partial x_2} + \frac{n^2}{k}\left(\frac{a}{3k} - k\mathcal{P}(x_1)\right)\frac{\partial\psi}{\partial x_2}$$
$$-\frac{n^2}{k}\left(\frac{a}{3k} - k\mathcal{P}(x_1)\right)\left(\zeta(x_1) + \frac{a}{3k^2}x_1\right)\frac{\partial\xi_1}{\partial x_3} = 0 \ .$$

Equating to zero the terms in $\mathcal{P}'(x_1)$, $\mathcal{P}(x_1)\zeta(x_1)$ leads to

$$k\frac{\partial\chi}{\partial x_2} = -\xi_1 \ , \ \frac{\partial\xi_1}{\partial x_3} = -\xi_1 \tag{115}$$

and subsequently

$$\frac{\partial\psi}{\partial x_2} = \frac{h - 2x_2}{2k}\,\xi_1 \ . \tag{116}$$

Taking into account the equations obtained so far, (113) become

$$\xi_2 = \frac{\alpha_0 x_2 - \beta_0}{n^2}\,\xi_1 + \psi(x_2, x_3) \ , \ \xi_3 = \frac{\alpha_0}{n^2}\xi_1 + \chi(x_2, x_3)$$

and so one has[60]

$$\frac{\partial\xi_2}{\partial x_2} = \left(\frac{\alpha_0}{n^2} + \frac{h - 2x_2}{2k}\right)\xi_1 \ , \ \frac{\partial\xi_2}{\partial x_3} = -\frac{\alpha_0 x_2 - \beta_0}{n^2}\,\xi_1 + \frac{\partial\psi}{\partial x_3} \ ,$$
$$\frac{\partial\xi_3}{\partial x_2} = \frac{\partial\chi}{\partial x_2} \ , \ \frac{\partial\xi_3}{\partial x_3} = -\frac{\alpha_0}{n^2}\xi_1 + \frac{\partial\chi}{\partial x_3} \ .$$

Substituting into (109), we then find

$$\frac{\partial\chi}{\partial x_2} = \frac{x_2}{k}\xi_1 \ , \ \frac{\partial\psi}{\partial x_3} = \psi + \left(\frac{x_2^2}{k} - \frac{hx_2}{k} - \frac{2a}{3k}\right)\xi_1$$

and finally we find for the most general values of ξ_1, ξ_2, ξ_3:[61]

$$\xi_1 = c_1 e^{-x_3} \ , \ \xi_2 = c_1\left\{\frac{\alpha_0 x_2 - \beta_0}{n^2} - \frac{x_2^2}{2k} + \frac{hx_2}{2k} + \frac{a}{3k}\right\}e^{-x_3} \ ,$$
$$\xi_3 = c_1\frac{\alpha_0}{n^2}e^{-x_3} - \frac{c_1 x_2}{k}e^{-x_3} + c_3 \ ,$$

with three arbitrary constants c_1, c_2, c_3. Therefore in the general case considered in the present section the complete group of motions is only a G_3.

[60] In the original paper, the second denominator on the r.h.s of the first equation is just k. Correction made after the *Opere* [Editor].

[61] The term $hx_2/2k$ in the second equation is absent in the original paper; correction made after the *Opere* [Editor].

31 Another method for the groups of type VIII.

In the work of the previous sections on the spaces which admit a transitive group G_3 of motions of type VIII we have seen the elliptical functions introduced. This depends on having wished to establish the geodesic form of the line element, making evident a family of pseudospherical surfaces, geodesically parallel and invariant with respect to a subgroup of two parameters. But, if we aim only to establish any form whatsoever for the line element, we can proceed much more directly by applying the general method described in §12 to a simple form of the group G_3. We now discuss this second way of treating the problem. In any event, we necessarily have to apply it in the last case of the groups of type IX, because there (real) 2-parameter subgroups do not exist.

We start from the theorem of Lie that two simply transitive and equally composed groups are always similar. Therefore if we take any particular form whatsoever of a group G_3 transitive over three variables with the composition of type VIII and determine the *most general* 3-dimensional spaces which admit it as a group of motions, any other space with a transitive group of motions of the same composition will necessarily be identical with one of these.

Given this, referring ourselves to the calculations made at the beginning of §28, we choose for the type of G_3 transitive over three variables of composition

$$[X_1, X_2]f = X_1 f \ , \ [X_1, X_3]f = 2X_2 f \ , \ [X_2, X_3]f = X_3 f$$

the one which is generated by the following three infinitesimal transformations:[62]

$$X_1 f = e^{-x_3}\frac{\partial f}{\partial x_1} - x_2^2 e^{-x_3}\frac{\partial f}{\partial x_2} - 2x_2 e^{-x_3}\frac{\partial f}{\partial x_3} \ ,$$
$$X_2 f = \frac{\partial f}{\partial x_3} \ , \ X_3 f = e^{x_3}\frac{\partial f}{\partial x_2}$$

and we determine, in the most general way, the coefficients of the line element of the space

$$ds^2 = \sum_{i,k} a_{ik}\, dx_i dx_k$$

so that it admits the group G_3.

For this we must make use of the fundamental equations (A), or equivalently (D) §12, applying them to the above three transformations. Beginning with $X_2 f$ we see that *the 6 coefficients have to be independent of* x_3. Then applying them to $X_3 f$, we find:

$$\frac{\partial a_{11}}{\partial x_2} = 0 \ , \ \frac{\partial a_{22}}{\partial x_2} = 0 \ , \ \frac{\partial a_{33}}{\partial x_2} + 2a_{23} = 0 \ ,$$
$$\frac{\partial a_{12}}{\partial x_2} = 0 \ , \ \frac{\partial a_{13}}{\partial x_2} + a_{12} = 0 \ , \ \frac{\partial a_{23}}{\partial x_2} + a_{22} = 0 \ ,$$

from which by integrating

$$a_{11} = A \ , \ a_{12} = B \ , \ a_{22} = C \ ,$$
$$a_{13} = D - Bx_2 \ , \ a_{23} = E - Cx_2 \ , \ a_{33} = Cx_2^2 - 2Ex_2 + F \ ,$$

[62] The original paper has $\frac{\partial f}{\partial x_3}$ instead of $\frac{\partial f}{\partial x_2}$ in the first equation; correction after the *Opere* [Editor].

where A, B, C, D, E, F are functions only of x_1. Finally if we apply them to $X_1 f$, taking into account the preceding values of the $\partial a_{ik}/\partial x_2$, we obtain

$$\frac{\partial a_{11}}{\partial x_1} = 0 \ , \ \frac{\partial a_{22}}{\partial x_1} = 4a_{22}x_2 + 4a_{23} \ ,$$
$$\frac{\partial a_{33}}{\partial x_1} = 2a_{13} - 4a_{23}x_2^2 - 4a_{33}x_2 \ ,$$
$$\frac{\partial a_{12}}{\partial x_1} = 2a_{12}x_2 + 2a_{13} \ , \ \frac{\partial a_{13}}{\partial x_1} = a_{11} - 2a_{12}x_2^2 - 2a_{13}x_2 \ ,$$
$$\frac{\partial a_{23}}{\partial x_1} = a_{12} + 2a_{33} - 2a_{22}x_2^2 \ ,$$

from which we derive

$$A = \mathit{constant} \ , \ C' = 4E \ , \ E' = B + 2F \ ,$$
$$F' = 2D \ , \ B' = 2D \ , \ D' = A \ ,$$

and so

$$\begin{aligned} &A = a^2 \ , \ B = a^2x_1^2 + 2bx_1 + c \ , \\ &C = a^2x_1^4 + 4bx_1^3 + 2(c+2d)x_1^2 + 4ex_1 + f \ , \ D = a^2x_1 + b \ , \\ &E = a^2x_1^3 + 3bx_1^2 + (c+2d)x_1 + e \ , \ F = a^2x_1^2 + 2bx_1 + d \ , \end{aligned} \tag{117}$$

with a, b, c, d, e, f six arbitrary constants. In conformity with the general theorem of §12, we verify in this way that our system of total differential equations is completely integrable, the initial values of the 6 coefficients a_{ik} remaining arbitrary for $x_1 = x_2 = 0$.

We observe that from equations (117) it follows that C is an arbitrary fourth degree polynomial in x_1; say $Q(x_1)$, with the first coefficient positive (or zero), and one then has[63]

$$A = \frac{Q^{(4)}(x_1)}{24} \ , \ B = \frac{Q''(x_1)}{12} + h \ , \ C = Q(x_1) \ ,$$
$$D = \frac{Q'''(x_1)}{24} \ , \ E = \frac{Q'(x_1)}{4} \ , \ F = \frac{Q''(x_1)}{12} - \frac{h}{2} \ ,$$

with h an arbitrary constant.

The surfaces invariant with respect to the subgroup $(X_2 f, X_3 f)$ are $x_1 = \mathit{constant}$; these are geodesically parallel, as follows from the general theorem and as we confirm here by calculating the differential parameter of the first order for x_1, which has the value

$$\Delta_1 x_1 = a_{22}a_{33} - a_{23}^2 = CF - E^2 \ .$$

From the expressions (117) for C, E, F, the binomial $CF - E^2$ is a fourth degree polynomial $P(x_1)$ in x_1. The arclength s of the geodesics orthogonal to the surfaces $x_1 = \mathit{constant}$ is given, as one knows, by

$$s = \int \frac{dx_1}{\sqrt{\Delta_1 x_1}} = \int \frac{dx_1}{\sqrt{P(x_1)}} \ ,$$

from which we again see the elliptic functions introduced here, confirming what we have said at the beginning of the present section.

[63]Bianchi's Q^{IV} was replaced by the more familiar $Q^{(4)}$ [Editor].

32 The groups of type IX:

$[X_1, X_2]f = X_3f \ , [X_2, X_3]f = X_1f \ , [X_3, X_1]f = X_2f$.

According to the method described at the beginning of the previous section, we must first choose here the form of a group G_3 transitive over three variables of the desired composition. We fix as the general type the group generated by the following three infinitesimal transformations:

$$\begin{aligned} X_1f &= \frac{\partial f}{\partial x_2} \ , \\ X_2f &= \cos x_2 \frac{\partial f}{\partial x_1} - \cot x_1 \sin x_2 \frac{\partial f}{\partial x_2} + \frac{\sin x_2}{\sin x_1}\frac{\partial f}{\partial x_3} \ , \\ X_3f &= -\sin x_2 \frac{\partial f}{\partial x_1} - \cot x_1 \cos x_2 \frac{\partial f}{\partial x_2} + \frac{\cos x_2}{\sin x_1}\frac{\partial f}{\partial x_3} \ , \end{aligned}$$

which is clearly transitive and offers the composition IX. If a space of line element

$$ds^2 = \sum_{i,k} a_{ik}\, dx_i dx_k$$

is to admit this group as a group of motions, first of all the coefficients a_{ik} must be independent of x_2 because of the form of X_1f. Secondly, expressing by means of (D) §12 the fact that the space admits the infinitesimal transformation X_2f (or the other X_3f)[64] we find the following system of partial differential equations[65] for the 6 coefficients a_{ik}:

$$\begin{aligned} &\frac{\partial a_{11}}{\partial x_1} = 0 \ , \ \frac{\partial a_{11}}{\partial x_3} = 2a_{13}\cot x_1 - \frac{2a_{12}}{\sin x_1} \ , \\ &\frac{\partial a_{22}}{\partial x_1} = 2a_{22}\cot x_1 - \frac{2a_{23}}{\sin x_1} \ , \ \frac{\partial a_{22}}{\partial x_3} = 2a_{12}\sin x_1 \ , \\ &\frac{\partial a_{33}}{\partial x_1} = 0 \ , \ \frac{\partial a_{33}}{\partial x_3} = 0 \ , \\ &\frac{\partial a_{12}}{\partial x_1} = 2a_{12}\cot x_1 - \frac{a_{13}}{\sin x_1} \ , \ \frac{\partial a_{12}}{\partial x_3} = a_{11}\sin x_1 + a_{23}\cot x_1 - \frac{a_{22}}{\sin x_1} \ , \\ &\frac{\partial a_{13}}{\partial x_1} = 0 \ , \ \frac{\partial a_{13}}{\partial x_3} = a_{33}\cot x_1 - \frac{a_{23}}{\sin x_1} \ , \\ &\frac{\partial a_{23}}{\partial x_1} = a_{23}\cot x_1 - \frac{a_{33}}{\sin x_1} \ , \ \frac{\partial a_{23}}{\partial x_3} = a_{13}\sin x_1 \ . \end{aligned}$$

We observe that a_{33} is a constant and so

$$\frac{\partial^2 a_{13}}{\partial {x_3}^2} = -\frac{1}{\sin x_1}\frac{\partial a_{23}}{\partial x_3} = -a_{13} \ ,$$

we then have

$$a_{33} = a^2 \ , \ a_{13} = b\cos x_3 + c\sin x_3 \ ,$$

with a, b, c constants. Substituting into the formula which gives $\partial a_{13}/\partial x_3$ we obtain

$$a_{23} = a^2\cos x_1 + \sin x_1\,(b\sin x_3 - c\cos x_3) \ .$$

[64] It suffices that it admit the two X_1f, X_2f in order for it to also admit the third since $[X_1, X_2]f = X_3f$.

[65] The original paper had a_{23} on the l.h.s. of the 4th equation and $\partial/\partial x_2$ on the l.h.s. of the 8th equation, both of which were incorrect [Editor].

Now since a_{11} is a function only of x_3, we set $a_{11} = 2\varphi(x_3)$, so that from the formula which gives $\partial a_{11}/\partial x_3$ it follows that

$$a_{12} = \cos x_1 \,(b\cos x_3 + c\sin x_3) - \sin x_1 \,\varphi'(x_3) \ .$$

Then integrating the two equations for a_{22} we have

$$\begin{aligned} a_{22} = {} & 2\sin x_1 \cos x_1 \,(b\sin x_3 - c\cos x_3) - 2\sin^2 x_1 \,\varphi(x_3) \\ & + a^2 + d\sin^2 x_1 \ , \end{aligned}$$

with d a new constant. Finally by substituting into the formula which gives $\partial a_{12}/\partial x_3$, we find for $\varphi(x_3)$ the differential equation

$$\varphi''(x_3) = -4\varphi(x_3) + a^2 + d \ ,$$

and so by integration[66]

$$\varphi(x_3) = e\cos(2x_3) + f\sin(2x_3) + (a^2 + d)/4 \ ,$$

with e, f new constants. With the values thus determined for the 6 coefficients a_{ik}, the above stated equations are actually satisfied, whatever the 6 constants a, b, c, d, e, f are.

We can then directly show, making use of the usual fundamental equations (A), that the complete group of motions is the given G_3, except when the four constants b, c, e, f are simultaneously zero. We prefer to treat this problem in another way, taking advantage of the theorem of Lie on the composition of groups, which makes the work simpler. We add that we can also apply the same method to the groups of type VIII to derive again the results of §28, §30.

33 Spaces which admit as a subgroup of motions a group G_3 of type IX.

Suppose that we have a space which admits a transitive G_3 of type IX as a subgroup of motions, but that its group of motions is larger. If we exclude the case of spaces of constant curvature, this larger group cannot be other than a 4-parameter group, a fact which we state here postponing its demonstration to §36.

Given the hypothesized G_4 containing the simple subgroup G_3 of composition[67]

$$[X_1, X_2]f = X_3 f \ , \ [X_2, X_3]f = X_1 f \ , \ [X_3, X_1]f = X_2 f \ ,$$

by the indicated theorem of Lie,[68] we can choose the fourth infinitesimal generating transformation of G_4 so that one has

$$[X_1, X_4]f = [X_2, X_4]f = [X_3, X_4]f = 0 \ .$$

[66] The original paper has $x_3/2$ instead of $2x_3$; correction after the *Opere* [Editor].

[67] The original paper had X_1, X_3 on the left in the first commutator, an obvious typo [Editor].

[68] See S. Lie-F. Engel, Vol. III, p. 723 and S. Lie-C. Scheffers, p. 574, Theorem 9. — It is worth noting that the theorems used here depend only on the relationships among the constants of composition c_{iks} and do not lose their validity by limiting them to the consideration of real groups and subgroups, as we do here.

We consider in G_4 the G_2 of Abelian motions generated for example by X_1f, X_4f and as in §14 we choose as coordinate surfaces $x_1 = \textit{constant}$ the surfaces invariant with respect to the group G_2. Proceeding as in the cited section we can furthermore assume $X_1f = \partial f/\partial x_2$, $X_4f = \partial f/\partial x_3$, and give the line element of the space the form

$$ds^2 = dx_1^2 + \alpha\, dx_2^2 + 2\beta\, dx_2 dx_3 + \gamma\, dx_3^2 \ ,$$

with α, β, γ functions only of x_1. Now let $X_2f = \eta_1\partial f/\partial x_1 + \eta_2\partial f/\partial x_2 + \eta_3\partial f/\partial x_3$; because of $[X_1, X_2]f = X_3f$, it follows that

$$X_3f = \frac{\partial\eta_1}{\partial x_2}\frac{\partial f}{\partial x_1} + \frac{\partial\eta_2}{\partial x_2}\frac{\partial f}{\partial x_2} + \frac{\partial\eta_3}{\partial x_2}\frac{\partial f}{\partial x_3} \ .$$

Since on the other hand one must also have $[X_2, X_4]f = 0$, $[X_3, X_1]f = X_2f$, η_1, η_2, η_3 must satisfy the conditions:

$$\begin{aligned}
&\frac{\partial\eta_1}{\partial x_3} = \frac{\partial\eta_2}{\partial x_3} = \frac{\partial\eta_3}{\partial x_3} = 0 \ , \\
&\frac{\partial^2\eta_1}{\partial {x_2}^2} + \eta_1 = 0 \ , \frac{\partial^2\eta_2}{\partial {x_2}^2} + \eta_2 = 0 \ , \frac{\partial^2\eta_3}{\partial {x_2}^2} + \eta_3 = 0 \ ,
\end{aligned}$$

from which we will have

$$\begin{aligned}
&\eta_1 = A\sin x_2 + B\cos x_2 \ , \ \eta_2 = C\sin x_2 + D\cos x_2 \ , \\
&\eta_3 = E\sin x_2 + F\cos x_2 \ ,
\end{aligned}$$

where A, D, C, D, E, F are functions only of x_1. From the first of equations (E) §14, it follows that η_1 does not depend on x_1 and so A, B are absolute constants. Finally from the composition equation $[X_2, X_3]f = X_1f$ we get the following three equations

$$\begin{aligned}
&AC + BD = 0 \ , \ BC' - AD' = C^2 + D^2 + 1 \ , \\
&AF' - BE' + CE + DF = 0 \ .
\end{aligned}$$

There is no loss of generality in adding a constant to x_2 in such a way that $B = 0$ [69] and since one cannot simultaneously have $A = 0$, as can be seen from the second of the equations (β), we will also have $C = 0$, so $-AD' = 1 + D^2$, $AF' + DF = 0$.

Integrating the first equation and ignoring the additive constant in x_1, as is allowed, we will have $D = -\tan(x_1/A)$, $F = k/\cos(x_1/A)$, with k an arbitrary constant.

If we now apply the other equations (E) §14, the relation

$$\frac{1}{2}\gamma'\eta_1 + \beta\frac{\partial\eta_2}{\partial x_3} + \gamma\frac{\partial\eta_3}{\partial x_3} = 0$$

shows us that γ is constant, so we set $\gamma = h^2$ and the remaining equations give us

$$\begin{aligned}
&E = 0 \ , \\
&\alpha D' + \beta F' = -A \ , \ \beta D' + h^2 F' = 0 \ , \\
&A\alpha'/2 = \alpha D + \beta F \ , \ A\beta' = \beta D + h^2 F \ ,
\end{aligned}$$

[69] This assumes $A \neq 0$; if this is not true, one would change x_2 into $\pi/2 + x_2$.

from which we get

$$\alpha = A^2 \cos\left(\frac{x_1}{A}\right) + h^2 k^2 \sin^2\left(\frac{x_1}{A}\right) \ , \ \beta = h^2 k \sin\left(\frac{x_1}{A}\right) \ , \ \gamma = h^2 \ .$$

If we set $x_1 = Ay_1$, $x_3 = Ay_3/h$, $n = hk/A$, then dividing the line element by h^2, we find the standard form

$$ds^2 = dy_1^2 + (\cos^2 y_1 + n^2 \sin^2 y_1)\, dx_2^2 + 2n \sin y_1\, dx_2 dy_3 + dy_3^2$$

which, by changing the notation, we can write as

$$ds^2 = dx_1^2 + (\sin^2 x_1 + n^2 \cos^2 x_1)\, dx_2^2 + 2n \cos x_1\, dx_2 dx_3 + dx_3^2 \ . \tag{118}$$

One sees that for $n = 0$ we obtain the space already considered in §9. This case must be excluded here though because it would lead to $\eta_3 = 0$ and the derived group $(X_1 f, X_2 f, X_3 f)$ is then intransitive.

We also exclude the case $n = 1$ because the line element then becomes

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + 2\cos x_1\, dx_2 dx_3$$

and belongs to the space of constant positive curvature $K = 1/4$. In fact let $x_1 = 2y_1$, $x_2 = y_2 + y_3$, $x_3 = y_2 - y_3$ and one has

$$ds^2 = 4(dy_1^2 + \cos^2 y_1\, dy_2^2 + \sin^2 y_1\, dy_3^2)$$

which indeed belongs to one such space. The geodesically parallel surfaces $x_1 = \textit{constant}$ are in this case Clifford surfaces of zero curvature.

34 The complete group of motions of the space: $ds^2 = dx_1^2 + (\sin^2 x_1 + n^2 \cos^2 x_1)\, dx_2^2 + 2n \cos x_1\, dx_2 dx_3 + dx_3^2$.

To determine the most general infinitesimal motion

$$Xf = \eta_1\, \partial f/\partial x_1 + \eta_2\, \partial f/\partial x_2 + \eta_3\, \partial f/\partial x_3$$

of the space of the line element (118), the equations (E) §14 give us the following equations

$$\frac{\partial \eta_1}{\partial x_1} = 0 \ , \tag{119}$$

$$\frac{\partial \eta_1}{\partial x_2} + (\sin^2 x_1 + n^2 \cos^2 x_1) \frac{\partial \eta_2}{\partial x_1} + n \cos x_1 \frac{\partial \eta_3}{\partial x_1} = 0 \ , \tag{120a}$$

$$\frac{\partial \eta_1}{\partial x_3} + n \cos x_1 \frac{\partial \eta_2}{\partial x_1} + \frac{\partial \eta_3}{\partial x_1} = 0 \ , \tag{120b}$$

$$(1 - n^2) \sin x_1 \cos x_1\, \eta_1 + (\sin^2 x_1 + n^2 \cos^2 x_1) \frac{\partial \eta_2}{\partial x_2} + n \cos x_1 \frac{\partial \eta_3}{\partial x_2} = 0 \ , \tag{121}$$

$$n \cos x_1 \frac{\partial \eta_2}{\partial x_3} + \frac{\partial \eta_3}{\partial x_3} = 0 \ , \tag{122}$$

$$-n \sin x_1\, \eta_1 + (\sin^2 x_1 + n^2 \cos^2 x_1) \frac{\partial \eta_2}{\partial x_3} + n \cos x_1 \left(\frac{\partial \eta_2}{\partial x_2} + \frac{\partial \eta_3}{\partial x_3} \right) + \frac{\partial \eta_3}{\partial x_2} = 0 \ . \tag{123}$$

Solving (120) for $\partial\eta_2/\partial x_1$, $\partial\eta_3/\partial x_1$, and integrating with respect to x_1, on which η_1 does not depend by (119), leads to the result:

$$\eta_2 = \cot x_1 \frac{\partial\eta_1}{\partial x_2} - \frac{n}{\sin x_1}\frac{\partial\eta_1}{\partial x_3} + \psi(x_2, x_3) \ , \tag{124a}$$

$$\eta_3 = \left\{n^2 \cot x_1 - (1-n^2)x_1\right\} \frac{\partial\eta_1}{\partial x_3} - \frac{n}{\sin x_1}\frac{\partial\eta_1}{\partial x_2} + \chi(x_2, x_3) \ . \tag{124b}$$

Substituting into (122), we have

$$-n\sin x_1 \frac{\partial^2\eta_1}{\partial x_2 \partial x_3} + n\cos x_1 \frac{\partial\psi}{\partial x_3} + (n^2-1)x_1 \frac{\partial^2\eta_1}{\partial {x_3}^2} + \frac{\partial\chi}{\partial x_2} = 0 \ ,$$

and so since $n^2 - 1 \neq 0$:

$$\frac{\partial^2\eta_1}{\partial x_2 \partial x_3} = \frac{\partial^2\eta_1}{\partial {x_3}^2} = 0 \ , \ \frac{\partial\psi}{\partial x_3} = \frac{\partial\chi}{\partial x_3} = 0 \ .$$

Substituting now into (121), we then obtain

$$\frac{\partial\psi}{\partial x_2} = \frac{\partial\chi}{\partial x_2} = 0 \ , \ \frac{\partial^2\eta_1}{\partial {x_2}^2} = -\eta_1 \ ,$$

and with these equations (123) is also satisfied. It follows next that $\partial\eta_1/\partial x_3 = 0$, so

$$\begin{aligned}
\eta_1 &= a\cos x_2 + b\sin x_2 \ , \\
\eta_2 &= \cot x_1(-a\sin x_2 + b\cos x_2) + c \ , \\
\eta_3 &= \frac{n}{\sin x_1}(a\sin x_2 - b\cos x_2) + d \ ,
\end{aligned}$$

with a, b, c, d arbitrary constants.

Therefore the complete group of motions of the space (118) is the G_4 generated by the four infinitesimal transformations:

$$\begin{aligned}
X_1 f &= \frac{\partial f}{\partial x_2} \ , \\
X_2 f &= \cos x_2 \frac{\partial f}{\partial x_1} - \cot x_1 \sin x_2 \frac{\partial f}{\partial x_2} + \frac{n\sin x_2}{\sin x_1}\frac{\partial f}{\partial x_3} \ , \\
X_3 f &= -\sin x_2 \frac{\partial f}{\partial x_1} - \cot x_1 \cos x_2 \frac{\partial f}{\partial x_2} + \frac{n\cos x_2}{\sin x_1}\frac{\partial f}{\partial x_3} \ , \\
X_4 f &= \frac{\partial f}{\partial x_3} \ ,
\end{aligned}$$

which has the composition:

$$\begin{aligned}
&[X_1, X_2]f = X_3 f \ , \ [X_2, X_3]f = X_1 f \ , \ [X_3, X_1]f = X_2 f \ , \\
&[X_1, X_4]f = [X_2, X_4]f = [X_3, X_4]f = 0 \ .
\end{aligned}$$

This group is systatic and the systatic varieties are the geodesics (x_3), which however, except in the case $n = 0$, do not admit orthogonal trajectories.

35 The constant n is essential in $ds^2 = dx_1^2 + (\sin^2 x_1 + n^2 \cos^2 x_1)\, dx_2^2 + 2n \cos x_1\, dx_2 dx_3 + dx_3^2$.

We wish to show finally that in the line element (113) the constant n, apart from sign,[70] is actually essential and namely that if a second space

$$ds^2 = dy_1^2 + (\sin^2 y_1 + m^2 \cos^2 y_1)\, dy_2^2 + 2m \cos y_1\, dy_2 dy_3 + dy_3^2 \tag{126}$$

is similar to the first, one must necessarily have $n^2 = m^2$.

Adopting for this the same method which has served us in the analogous cases, we observe that the group Γ_4 of motions of the space (126) is generated by the four infinitesimal transformations:

$$\begin{aligned}
Y_1 f &= \frac{\partial f}{\partial y_2} \ , \\
Y_2 f &= \cos y_2 \frac{\partial f}{\partial y_1} - \cot y_1 \sin y_2 \frac{\partial f}{\partial y_2} + \frac{m \sin y_2}{\sin y_1} \frac{\partial f}{\partial y_3} \ , \\
Y_3 f &= -\sin y_2 \frac{\partial f}{\partial y_1} - \cot y_1 \cos y_2 \frac{\partial f}{\partial y_2} + \frac{m \cos y_2}{\sin y_1} \frac{\partial f}{\partial y_3} \ , \\
Y_4 f &= \frac{\partial f}{\partial y_3} \ ,
\end{aligned}$$

with the composition:

$$\begin{aligned}
&[Y_1, Y_2]f = Y_3 f \ , \ [Y_2, Y_3]f = Y_1 f \ , \ [Y_3, Y_1]f = Y_2 f \ , \\
&[Y_1, Y_4]f = [Y_2, Y_4]f = [Y_3, Y_4]f = 0 \ .
\end{aligned}$$

First we must determine if the group G_4 of the first space is similar to the Γ_4 of the second. Assuming that the equations of transformation change $X_1 f$, $X_2 f$, $X_3 f$, $X_4 f$, respectively into $\bar{Y}_1 f$, $\bar{Y}_2 f$, $\bar{Y}_3 f$, $\bar{Y}_4 f$, the $\bar{Y} f$ must be combinations of the $Y f$ and have their same composition. From this it follows that, since $Y_4 f$ is the only infinitesimal transformation of Γ_4 which commutes with every other, $\bar{Y}_4 f$ will not differ from it other than by a constant factor a, while $\bar{Y}_1 f$, $\bar{Y}_2 f$, $\bar{Y}_3 f$, belonging to the derived group, will not involve $Y_4 f$ and one will have[71]

$$\begin{aligned}
\bar{Y}_1 f &= c_{11} Y_1 f + c_{12} Y_2 f + c_{13} Y_3 f \ , \\
\bar{Y}_2 f &= c_{21} Y_1 f + c_{22} Y_2 f + c_{23} Y_3 f \ , \\
\bar{Y}_3 f &= c_{31} Y_1 f + c_{32} Y_2 f + c_{33} Y_3 f \ , \ \bar{Y}_4 f = a Y_4 f \ .
\end{aligned}$$

The composition equations

$$[\bar{Y}_1, \bar{Y}_2]f = \bar{Y}_3 f \ , \ [\bar{Y}_2, \bar{Y}_3]f = \bar{Y}_1 f \ , \ [\bar{Y}_3, \bar{Y}_1]f = \bar{Y}_2 f$$

show that the nine constants c_{ik} are the coefficients of an orthogonal matrix of determinant $= +1$. Now among $X_1 f$, $X_2 f$, $X_3 f$, $X_4 f$, holds the unique relation

$$X_4 f = \frac{1}{n} \{\cos x_1\, X_1 f + \sin x_1 \sin x_2\, X_2 f + \sin x_1 \cos x_2\, X_3 f\}$$

[70] Changing the sign of either x_2 or x_3 changes the sign of n.

[71] The original paper had X_2 instead of Y_2 in the second equation, an obvious typo [Editor].

and similarly among the $Y_i f$ the other

$$Y_4 f = \frac{1}{m}\{\cos y_1\, Y_1 f + \sin y_1\, \sin y_2 Y_2 f + \sin y_1 \cos y_2\, Y_3 f\} \ .$$

Expressing the x in terms of the y, we have

$$\bar{Y}_4 f = \frac{1}{n}\{\cos x_1\, \bar{Y}_1 f + \sin x_1 \sin x_2\, \bar{Y}_2 f + \sin x_1 \cos x_2\, \bar{Y}_3 f\}$$

or equivalently

$$\begin{aligned} Y_4 = \frac{1}{an}\{&(c_{11}\cos x_1 + c_{21}\sin x_1 \sin x_2 + c_{31}\sin x_1 \cos x_2) Y_1 f \\ &+(c_{12}\cos x_1 + c_{22}\sin x_1 \sin x_2 + c_{32}\sin x_1 \cos x_2) Y_2 f \\ &+(c_{13}\cos x_1 + c_{23}\sin x_1 \sin x_2 + c_{33}\sin x_1 \cos x_2) Y_3 f\} \ . \end{aligned}$$

Comparing this with (γ) leads to the three equations

$$\begin{aligned} (an/m)\cos y_1 &= c_{11}\cos x_1 + c_{21}\sin x_1 \sin x_2 + c_{31}\sin x_1 \cos x_2 \ , \\ (an/m)\sin y_1 \sin y_2 &= c_{12}\cos x_1 + c_{22}\sin x_1 \sin x_2 + c_{32}\sin x_1 \cos x_2 \ , \\ (an/m)\sin y_1 \cos y_2 &= c_{13}\cos x_1 + c_{23}\sin x_1 \sin x_2 + c_{33}\sin x_1 \cos x_2 \ . \end{aligned}$$

The compatibility of these three equations in y_1, y_2 gives, according to the general theory, the necessary and sufficient condition for the similarity of the two groups G_4, Γ_4. This condition is found immediately by squaring and summing the above three equations, which gives $a^2 n^2/m^2 = 1$. It suffices therefore to take $a = \pm m/n$ in order that corresponding equations of transformation of G_4 into Γ_4 exist. Equations (127) show that one has $\partial y_1/\partial x_3 = \partial y_2/\partial x_3 = 0$.

For the rest, expressing the fact that the $X_i f$ are transformed respectively into the $\bar{Y}_i f$, we can find all the values of the first partial derivatives of the y with respect to the x. It suffices for us to note here, in addition to the two above, the following

$$\frac{\partial y_2}{\partial x_2} = c_{11} - c_{12}\cot y_1 \sin y_2 - c_{13}\cot y_1 \cos y_2 \ , \tag{128a}$$

$$\cos x_2 \frac{\partial y_1}{\partial x_1} = c_{22}\cos y_2 - c_{23}\sin y_2 + \cot x_1 \sin x_2 (c_{12}\cos y_2 - c_{13}\sin y_2) \ , \tag{128b}$$

$$\frac{\partial y_3}{\partial x_3} = a \ . \tag{128c}$$

Assuming now that the two line elements are transformable one into the other, except for a constant factor, we utilize as in §19 the Christoffel formula

$$\frac{\partial^2 y_\nu}{\partial x_r \partial x_s} + \sum_{i,k} \left\{ \begin{matrix} \nu \\ ik \end{matrix} \right\}_y \frac{\partial y_i}{\partial x_r}\frac{\partial y_k}{\partial x_s} = \sum_\mu \left\{ \begin{matrix} \mu \\ rs \end{matrix} \right\}_x \frac{\partial y_\nu}{\partial x_\mu} \ ,$$

setting $\nu = 1$, $r = 2$, $s = 3$ and substituting for the Christoffel symbols their actual values, we obtain

$$\sin y_1 \frac{\partial y_2}{\partial x_2} = \frac{n}{am}\sin x_1 \frac{\partial y_1}{\partial x_1} \ ,$$

or equivalently by (128)

$$\begin{aligned}&\cos x_2(c_{11}\sin y_1 - c_{12}\cos y_1\sin y_2 - c_{13}\cos y_1\cos y_2)\\&= \frac{n}{am}\{\sin x_1(c_{22}\cos y_2 - c_{23}\sin y_2)\\&\quad + \cos x_1\sin x_1(c_{12}\cos y_2 - c_{13}\sin y_2)\}\ .\end{aligned}$$

Multiplying this last equation by $a^2n^2/m^2\ \sin y_1 = \sin y_1$, noting (127) one obtains the equation

$$\begin{aligned}&c_{11}\cos x_2\left\{1-(c_{11}\cos x_1 + c_{21}\sin x_1\sin x_2 + c_{31}\sin x_1\cos x_2)^2\right\}\\&-c_{12}\cos x_2(c_{11}\cos x_1 + c_{21}\sin x_1\sin x_2 + c_{31}\sin x_1\cos x_2)\\&\qquad\times(c_{12}\cos x_1 + c_{22}\sin x_1\sin x_2 + c_{32}\sin x_1\cos x_2)\\&-c_{13}\cos x_2(c_{11}\cos x_1 + c_{21}\sin x_1\sin x_2 + c_{31}\sin x_1\cos x_2)\\&\qquad\times(c_{13}\cos x_1 + c_{23}\sin x_1\sin x_2 + c_{33}\sin x_1\cos x_2)\\&=\frac{n^2}{m^2}\{c_{22}\sin x_1(c_{13}\cos x_1 + c_{23}\sin x_1\sin x_2 + c_{33}\sin x_1\cos x_2)\\&+c_{12}\cos x_1\sin x_2(c_{13}\cos x_1 + c_{23}\sin x_1\sin x_2 + c_{33}\sin x_1\cos x_2)\\&-c_{23}\sin x_1(c_{12}\cos x_1 + c_{22}\sin x_1\sin x_2 + c_{32}\sin x_1\cos x_2)\\&-c_{13}\cos x_1\sin x_2(c_{12}\cos x_1 + c_{22}\sin x_1\sin x_2 + c_{32}\sin x_1\cos x_2)\}\ ,\end{aligned}$$

which must prove to be an identity in x_1, x_2. Setting $x_2 = 0$ in this equation we find

$$\frac{n^2}{m^2}(c_{11}\sin^2 x_1 - c_{31}\sin x_1\cos x_1) = c_{11}\sin^2 x_1 - c_{31}\sin x_1\cos x_1$$

so $n^2 = m^2$, unless one has $c_{11} = 0$, $c_{31} = 0$ so that also $c_{22} = 0$, $c_{23} = 0$, $c_{21} = \pm 1$.

Introducing these values of c into the above identity leads to: $(n^2/m^2 - 1)c_{21} = 0$ and so again $n^2 = m^2$, Q.E.D.

36 The impossibility of other spaces with continuous groups of motions.

In the previous sections we have exhausted the study of the 3-dimensional spaces which admit intransitive groups of motion or transitive 3-parameter groups. And now we show that with this we have also determined all the possible spaces which admit continuous groups of motions.

Therefore, since the group of motions of a space cannot have more than 6 parameters, it will clearly suffice to show that a (transitive) group of motions with 6, 5, or 4 parameters necessarily contains some *real* 3-parameter subgroup.

If we treat a G_6 this is clear since then the motions which leave a point of the space fixed form precisely a real subgroup with $6 - 3 = 3$ parameters.[72]

If the complete group of motions is a G_4 we easily find the same thing recalling that the derived group of a G_4 possesses at most 3 parameters and therefore, in any case, there exist real 3-parameter subgroups in G_4. And indeed if the G_4 is generated by the four infinitesimal transformations $X_1 f$, $X_2 f$, $X_3 f$, $X_4 f$, and the derived group is the identity,

[72] S. Lie-F. Engel, Vol. I, p. 204.

or (X_1f) or (X_1f, X_2f) or even (X_1f, X_2f, X_3f), then (X_1f, X_2f, X_3f) will always be a real 3-parameter subgroup.

It remains to show the same property for a G_5. In this transitive group those motions which leave an arbitrary point of the space fixed form a real G_2 and we propose to establish that such a G_2 would necessarily be contained in a real subgroup G_3 of the G_5. Lie[73] shows that indeed every G_2 in a group with $r \geq 3$ parameters is contained in at least one subgroup G_3; however, it could easily happen that in the general case these subgroups G_3 are only complex. But if we apply the same derivation given by Lie (*ibid.*) we see that our assertion will be proved when it is shown that if

$$E_1f = \sum_{i,k} a_{ik}x_k\frac{\partial f}{\partial x_i}\ ,\ E_2f = \sum_{i,k} b_{ik}x_k\frac{\partial f}{\partial x_i}$$

are two linear homogeneous transformations in three variables x_1, x_2, x_3 such that one has $[E_1, E_2]f = kE_1f$ (k constant) and one interprets x_1, x_2, x_3 as homogeneous coordinates of a point in a plane, then there will be at least one *real* point that will remain fixed by both transformations (fixed point). It is known that to find the fixed points with respect to the E_1f one has the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= \rho x_1\ , \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= \rho x_2\ , \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= \rho x_3 \end{aligned}$$

and since the cubic equation with real coefficients

$$\begin{vmatrix} a_{11}-\rho & a_{12} & a_{13} \\ a_{21} & a_{22}-\rho & a_{23} \\ a_{31}-\rho & a_{32} & a_{33}-\rho \end{vmatrix} = 0$$

has at least one real root, there will certainly be at least one real fixed point with respect to E_1f. If there exists for E_1 an *isolated* real fixed point, then since by assumption $[E_1, E_2]f = kE_1f$, it will be fixed with respect to E_2f.[74] So it will suffice to consider the case in which E_1f has no real isolated fixed points. This happens only when the above cubic equation has a single root, which furthermore makes all the second order minors of the same determinant zero.[75] Then all the fixed points are distributed over a (real) line and if we assume this line as the side $x_3 = 0$ of the fundamental triangle, we give to E_1f, as is immediately seen, the form[76]

$$E_1f = \rho\left(x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + x_3\frac{\partial f}{\partial x_3}\right) + \alpha x_3\frac{\partial f}{\partial x_1} + \beta x_3\frac{\partial f}{\partial x_2}\ .$$

If one had[77] $\alpha = \beta = 0$, E_1f would leave every point fixed and a real fixed point of E_2f would satisfy the required condition. If $\beta \neq 0$, changing x_1 into $x_1 + hx_2$, we can make

[73]S. Lie-F. Engel, Vol. I, pp.592–593.

[74]See S. Lie-F. Engel, Vol. 1, p. 507, Theor. 104.

[75]See the precise discussion in S. Lie-G. Scheffers, pp.510-511.

[76]The original paper had $\partial f/\partial x_3$ in the second term, an obvious typo [Editor].

[77]The original paper had $\alpha_3 = \beta_3 = 0$ here, an obvious typo [Editor].

$\alpha = 0$ and we will thus have

$$E_1 f = \rho \left(x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + x_3 \frac{\partial f}{\partial x_3} \right) + \beta x_3 \frac{\partial f}{\partial x_2} \ .$$

If $E_1 f, E_2 f$ were in the involution relation[78] $[E_1, E_2] f = 0$, the above considerations are already sufficient to demonstrate the assertion, since in the most unfavorable case where neither $E_1 f$ nor $E_2 f$ possess a real isolated fixed point, the meeting point of the two lines of the respective invariant points would satisfy the desired condition.

Therefore we assume in

$$[E_1, E_2] f = k E_1 f \qquad (a)$$

that $k \neq 0$. One then has[79]

$$E_2 f = (a_1 x_1 + a_2 x_2 + a_3 x_3) \frac{\partial f}{\partial x_1} + (b_1 x_1 + b_2 x_2 + b_3 x_3) \frac{\partial f}{\partial x_2} + (c_1 x_1 + c_2 x_2 + c_3 x_3) \frac{\partial f}{\partial x_3} \ .$$

The condition (a) gives $k\rho = 0$, $a_2 = c_1 = c_2 = 0$, $b_2 = c_3 + k$, so that $\rho = 0$ and we can make

$$E_1 f = x_3 \frac{\partial f}{\partial x_2} \ ,$$
$$E_2 f = (a_1 x_1 + a_3 x_3) \frac{\partial f}{\partial x_1} + (b_1 x_1 + b_2 x_2) \frac{\partial f}{\partial x_2} + c_3 x_3 \frac{\partial f}{\partial x_3} \ .$$

The real point of coordinates $(0, 1, 0)$ remains fixed by both transformations.

37 The impossibility of groups G_5 of motions.

By what we have shown in the previous section, there does not exist any space which has a G_5 for the complete group of motions. From this it follows that if a space should admit a subgroup G_5 of motions, also admitting a G_6, it would be of constant curvature. But we can easily go farther and show that the groups G_6 of motions of the spaces of constant curvature do not contain a real subgroups of 5 parameters, namely:

There does not exist any 3-dimensional space whose group of motions contains a real 5-parameter subgroup.

Assuming the existence of such a G_5 of motions, its subgroup G_2 which leaves any point P whatsoever of the space fixed is contained, by the previous section, in a real G_3. This G_3 would necessarily be transitive since otherwise with motions of G_3 one could transport P anywhere, but every point would remain fixed by a double infinity of motions of the G_3 which is absurd. The group G_3 being transitive, we can apply the methods developed in §§5–11 and therefore give the line element of the space of constant curvature one of the following 6 forms:

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \ , \ K = 0 \ , \qquad (\alpha 1)$$

[78] In Italian: "relazione involutoria" [Translator].

[79] The original paper had $\partial f / \partial x_1$ instead of $\partial f / \partial x_3$ in the last term [Editor].

$$ds^2 = dx_1^2 + e^{2x_1}(dx_2^2 + dx_3^2)\ ,\ K = -1\ , \tag{$\alpha2$}$$
$$ds^2 = dx_1^2 + x_1^2(dx_2^2 + \sin^2 x_2\, dx_3^2)\ ,\ K = 0\ , \tag{$\beta1$}$$
$$ds^2 = dx_1^2 + \sin^2 x_1(dx_2^2 + \sin^2 x_2\, dx_3^2)\ ,\ K = 1\ , \tag{$\beta2$}$$
$$ds^2 = dx_1^2 + \sinh^2 x_1(dx_2^2 + \sin^2 x_2\, dx_3^2)\ ,\ K = -1\ , \tag{$\beta3$}$$
$$ds^2 = dx_1^2 + \cosh^2 x_1(dx_2^2 + \sin^2 x_2\, dx_3^2)\ ,\ K = -1\ , \tag{γ}$$

which is adapted to the subgroup G_3 of rotations about a point[80] generated in the respective cases (α), (β), (γ) by the three infinitesimal transformations designated in §5 by (α^*), (β^*), (γ^*). For each of these forms we have to determine, by integrating the fundamental equations (A), the form of the complete group G_6 of motions and see if there exists a subgroup G_5 of the G_6 containing the G_3. The answer being negative, the stated property will be established.

Here I limit myself to carrying out the calculations for one case. We choose, for example, the (parabolic) form

$$ds^2 = dx_1^2 + e^{2x_1}(dx_2^2 + dx_3^2)$$

of the line element of the pseudospherical spaces. Integrating the equations of §7 we easily find that the complete group G_6 of motions is generated by the 6 infinitesimal transformations:

$$\begin{aligned}
&X_1 f = \frac{\partial f}{\partial x_2}\ ,\ X_2 f = \frac{\partial f}{\partial x_3}\ ,\ X_3 f = x_3\frac{\partial f}{\partial x_2} - x_2\frac{\partial f}{\partial x_3}\ ,\\
&X_4 f = x_2\frac{\partial f}{\partial x_1} + \frac{1}{2}(e^{-2x_1} + x_3^2 - x_2^2)\frac{\partial f}{\partial x_2} - x_2 x_3\frac{\partial f}{\partial x_3}\ ,\\
&X_5 f = x_3\frac{\partial f}{\partial x_1} - x_2 x_3\frac{\partial f}{\partial x_2} + \frac{1}{2}(e^{-2x_1} + x_2^2 - x_3^2)\frac{\partial f}{\partial x_3}\ ,\\
&X_6 f = \frac{\partial f}{\partial x_1} - x_2\frac{\partial f}{\partial x_2} - x_3\frac{\partial f}{\partial x_3}\ .
\end{aligned}$$

We now write the related composition equations:

$$\begin{aligned}
&[X_1, X_2]f = 0\ ,\ [X_1, X_3]f = -X_2 f\ ,\ [X_1, X_4]f = X_6 f\ ,\\
&\qquad [X_1, X_5]f = -X_3 f\ , [X_1, X_6]f = -X_1 f\ ,\\
&[X_2, X_3]f = X_1 f\ ,\ [X_2, X_4]f = X_3 f\ ,\ [X_2, X_5]f = X_6 f\ ,\\
&\qquad [X_2, X_6]f = -X_2 f\ ,\\
&[X_3, X_4]f = X_5 f\ ,\ [X_3, X_5]f = -X_4 f\ ,\ [X_3, X_6]f = 0\ ,\\
&[X_4, X_5]f = 0\ ,\ [X_4, X_6]f = X_4 f\ ,\\
&[X_5, X_6]f = X_5 f\ ,
\end{aligned}$$

the inspection of which would suffice to show us that there does not exist in the G_6 any real G_5 containing the subgroup $G_3 = (X_1 f, X_2 f, X_3 f)$.

[80] As is seen, in the space of zero curvature (Euclidean), we have two different forms for the line element, one (α) corresponding to the case of a center of rotation at infinity, the second (β) to the case of the center of rotation at a finite distance. For the pseudospherical spaces ($K = -1$) we have three distinct forms (α), (β), (γ), according to whether the center of rotation is at infinity or a finite distance, or is ideal, and finally for the space of Riemann ($K = +1$) only one form. These geometric circumstances are well known from the theory of spaces of constant curvature.

In fact let Yf be an infinitesimal transformation of G_5 that does not belong to G_3; we can set $Yf = aX_4f + bX_5f + cX_6f$ with a, b, c constants. In G_5 there will therefore also exist the three infinitesimal transformations

$$[X_1, Y]f = aX_6f - bX_3f - cX_1f \ ,$$
$$[X_2, Y]f = aX_3f + bX_6f - cX_2f \ ,$$
$$[X_3, Y]f = aX_5f - bX_4f \ ,$$

and so also aX_6f, bX_6f, and hence in any case X_6f since if $a = b = 0$, Yf reduces to X_6f. Now the four transformations X_1f, X_2f, X_3f, X_6f, of G_5 actually generate a G_4 and if by $Zf = aX_4f + bX_5f$ we indicate the last infinitesimal transformation, then $[X_3, Z]f = aX_5f - bX_4f$ must be a combination of X_1f, X_2f, X_3f, X_6f, Zf and so differs from Zf only by a constant factor ρ. Therefore one will have $a = \rho b$, $b = -\rho a$ from which $\rho^2 + 1 = 0$ and so $Zf = X_4f + iX_5f$, which gives only a complex G_5. Demonstrations completely analogous, as the reader can verify, are valid in all the other cases.

38 Summarized table of the line elements.

It will be useful to summarize the results obtained by gathering together in a table the various types to which we have reduced, in the course of this study, the line elements of all possible spaces which admit continous groups of motions.

We divide these spaces into six categories according to the type of their complete group G of motions. We assign a space to the category A) when its group of motions is a G_1, to B) when it is a G_2, to C) when it is an *intransitive* G_3. The other two categories D) , E) contain the spaces whose group of motions is *transitive*, D) those with a G_3, E) those with a G_4. Finally the sixth category F) will include the spaces of constant curvature which admit a group G_6 of motions. In the same table we also give the infinitesimal transformation generators and their composition.

Category A

Groups G_1

$ds^2 = \Sigma\, a_{ik}\, dx_i dx_k$

with coefficients a_{ik} independent of x_1

group:

$$X_1f = \frac{\partial f}{\partial x_1}$$

Category B

Groups G_2

$ds^2 = dx_1^2 + \alpha\, dx_2^2 + 2\beta\, dx_2 dx_3 + \gamma\, dx_3^2$

with α, β, γ functions only of x_1

group:

$$X_1 f = \frac{\partial f}{\partial x_2} \ , \ X_2 f = \frac{\partial f}{\partial x_3}$$

composition:

$$[X_1, X_2] = 0$$

$ds^2 = dx_1^2 + \alpha\, dx_2^2 + 2(\beta - \alpha x_2)\, dx_2 dx_3 + (\alpha x_2^2 - 2\beta x_2 + \gamma)\, dx_3^2$
with α, β, γ functions of x_1

group:

$$X_1 f = \frac{\partial f}{\partial x_3} \ , \ X_2 f = e^{x_3} \frac{\partial f}{\partial x_2}$$

composition:

$$[X_1, X_2] f = X_2 f$$

Category C

Groups G_3 intransitive

α) $\quad ds^2 = dx_1^2 + \varphi^2(x_1)(dx_2^2 + dx_3^2)$

with $\varphi(x_1)$ an arbitrary function of x_1

group:

$$X_1 f = \frac{\partial f}{\partial x_2} \ , \ X_2 f = \frac{\partial f}{\partial x_3} \ , \ X_3 f = x_3 \frac{\partial f}{\partial x_2} - x_2 \frac{\partial f}{\partial x_3}$$

composition:

$$[X_1, X_2] f = 0 \ , \ [X_1, X_3] f = -X_2 f \ , \ [X_2, X_3] f = X_1 f$$

β) $\quad ds^2 = dx_1^2 + \varphi^2(x_1)(dx_2^2 + \sin^2 x_2\, dx_3)^2$

group:

$$X_1 f = \frac{\partial f}{\partial x_3} \ , \ X_2 f = \sin x_3 \frac{\partial f}{\partial x_2} + \cot x_2 \cos x_3 \frac{\partial f}{\partial x_3} \ ,$$
$$X_3 f = \cos x_3 \frac{\partial f}{\partial x_2} - \cot x_2 \sin x_3 \frac{\partial f}{\partial x_3}$$

composition:

$$[X_1, X_2] f = X_3 f \ , \ [X_2, X_3] f = X_1 f \ , \ [X_3, X_1] f = X_2 f$$

γ) $\quad ds^2 = dx_1^2 + \varphi^2(x_1)\,(dx_2^2 + e^{2x_2}\, dx_3)^2$

group:

$$X_1 f = \frac{\partial f}{\partial x_3}\ ,\ X_2 f = \frac{\partial f}{\partial x_2} - x_3 \frac{\partial f}{\partial x_3}\ ,$$
$$X_3 f = x_3 \frac{\partial f}{\partial x_2} + \frac{1}{2}(e^{-2x_2} - x_3^2)\frac{\partial f}{\partial x_3}$$

composition:

$$[X_1, X_2]f = -X_1 f\ ,\ [X_1, X_3]f = X_2 f\ ,\ [X_2, X_3]f = -X_3 f$$

Category D

Groups G_3 transitive

Type IV
$ds^2 = dx_1^2 + e^{x_1}[dx_2^2 + 2x_1\, dx_2 dx_3 + (x_1^2 + n^2)\, dx_3)^2]$

group:

$$X_1 f = 2\frac{\partial f}{\partial x_2}\ ,\ X_2 f = \frac{\partial f}{\partial x_3}\ ,$$
$$X_3 f = -2\frac{\partial f}{\partial x_1} + (x_2 + 2x_3)\frac{\partial f}{\partial x_2} + x_3 \frac{\partial f}{\partial x_3}$$

composition:

$$[X_1, X_2] = 0\ ,\ [X_1, X_3]f = X_1 f\ ,\ [X_2, X_3]f = X_1 f + X_2 f$$

Type VI
$ds^2 = dx_1^2 + e^{2x_1}\, dx_2^2 + 2ne^{(h+1)x_1}\, dx_2 dx_3 + e^{2hx_1}\, dx_3^2$

group:

$$X_1 f = \frac{\partial f}{\partial x_2}\ ,\ X_2 f = \frac{\partial f}{\partial x_3}\ ,$$
$$X_3 f = -\frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + hx_3 \frac{\partial f}{\partial x_3}$$

composition:

$$[X_1, X_2] = 0\ ,\ [X_1, X_3]f = X_1 f\ ,\ [X_2, X_3]f = hX_2 f$$

Type VII
$$ds^2 = dx_1^2 + e^{-hx_1}\left\{(n + \cos vx_1)\, dx_2^2 + (h\cos vx_1 + v\sin x_1 + hn)\, dx_2 dx_3 + \left(\tfrac{2-v^2}{2}\cos vx_1 + \tfrac{hv}{2}\sin vx_1 + n\right)\, dx_3^2\right\}$$

group:

$$X_1 f = \frac{\partial f}{\partial x_2}\ ,\ X_2 f = \frac{\partial f}{\partial x_3}\ ,$$
$$X_3 f = \frac{\partial f}{\partial x_1} - x_3 \frac{\partial f}{\partial x_2} + (x_2 + hx_3)\frac{\partial f}{\partial x_3}$$

composition:

$$[X_1, X_2] = 0\ ,\ [X_1, X_3]f = X_2 f\ ,\ [X_2, X_3]f = -X_1 f + hX_2 f$$

Type VIII[81]
$ds^2 = \frac{Q^{(4)}(x_1)}{24}\,dx_1^2 + Q(x_1)\,dx_2^2 + \left(Q(x_1)x_2^2 - \frac{Q'(x_1)}{2}x_2 + \frac{Q''(x_1)}{2} - \frac{h}{2}\right)\,dx_3^2$
$+2\left(\frac{Q''(x_1)}{12} + h\right)\,dx_1dx_2 + 2\left\{\frac{Q'''(x_1)}{24} - \left(\frac{Q''(x_1)}{12} + h\right)x_2\right\}\,dx_1dx_3$
$+2\left(\frac{Q'(x_1)}{4} - Q(x_1)x_2\right)\,dx_2dx_3\ ,$

with $Q(x_1)$ a fourth degree polynomial in x_1 with its first coefficient positive (or zero), and h a constant

group:

$$X_1f = e^{-x_3}\frac{\partial f}{\partial x_1} - x_2^2e^{-x_3}\frac{\partial f}{\partial x_2} - 2x_2e^{-x_3}\frac{\partial f}{\partial x_3}\ ;$$
$$X_2f = \frac{\partial f}{\partial x_3}\ ,\ X_3f = \frac{\partial f}{\partial x_2}$$

composition:

$$[X_1, X_2]f = X_1f\ ,\ [X_1, X_3]f = 2X_2f\ ,\ [X_2, X_3]f = X_3f$$

Type IX[82]
$ds^2 = \Sigma_{i,k}\,a_{ik}\,dx_idx_k$

$$a_{11} = 2e\cos 2x_3 + 2f\sin 2x_3 + (a^2 + d^2)/2\ ,$$
$$a_{22} = 2\sin x_1\cos x_1(b\sin x_3 - c\cos x_3) - a_{11}\sin^2 x_1 + a^2 + d\sin^2 x_1\ ,$$
$$a_{33} = a^2\ ,\ a_{13} = b\cos x_3 + c\sin x_3\ ,$$
$$a_{12} = \cos x_1(b\cos x_3 + c\sin x_3) + 2\sin x_1(e\sin 2x_3 - f\cos 2x_3)\ ,$$
$$a_{23} = a^2\cos x_1 + \sin x_1(b\sin x_3 - c\cos x_3)$$

group:

$$X_1f = \frac{\partial f}{\partial x_2}\ ,\ X_2f = \cos x_2\frac{\partial f}{\partial x_1} - \cot x_1\sin x_2\frac{\partial f}{\partial x_2} + \frac{\sin x_2}{\sin x_1}\frac{\partial f}{\partial x_3}\ ,$$
$$X_3f = -\sin x_2\frac{\partial f}{\partial x_1} - \cot x_1\cos x_2\frac{\partial f}{\partial x_2} + \frac{\cos x_2}{\sin x_1}\frac{\partial f}{\partial x_3}$$

composition[83]:

$$[X_1, X_2]f = X_3f\ ,\ [X_2, X_3]f = X_1f\ ,\ [X_3, X_1]f = X_2f$$

Category E

Groups G_4 [84]

a) *[Type II]*
$ds^2 = dx_1^2 + dx_2^2 + 2x_1\,dx_2dx_3 + (x_1^2 + 1)\,dx_3^2$

[81]Bianchi's Q^{IV} was replaced by the more familiar $Q^{(4)}$ [Editor].
[82]The original paper had $x_3/2$ instead of $2x_3$, which was incorrect. Also, the second term in a_{12} had the coefficient $1/2$ instead of 2, corrected here after the *Opere* [Editor].
[83]The original paper had Xf in the second commutator on the r.h.s., an obvious typo [Editor].
[84]Simply transitive subgroup Bianchi types added in brackets by translator for clarity.

group:

$$X_1 f = \frac{\partial f}{\partial x_2}\ ,\ X_2 f = \frac{\partial f}{\partial x_3}\ ,\ X_3 f = -\frac{\partial f}{\partial x_1} + x_3\frac{\partial f}{\partial x_2}\ ,$$
$$X_4 f = x_3\frac{\partial f}{\partial x_1} + \frac{1}{2}(x_1^2 - x_3^2)\frac{\partial f}{\partial x_2} - x_1\frac{\partial f}{\partial x_3}$$

composition:

$$[X_1, X_2] = [X_1, X_3] = [X_1, X_4] = 0\ ,$$
$$[X_2, X_3]f = X_1 f\ ,\ [X_2, X_4]f = -X_3 f\ ,\ [X_3, X_4]f = X_2 f$$

b) *[Types III, VIII]*
$ds^2 = dx_1^2 + e^{2x_1}\,dx_2^2 + 2ne^{x_1}\,dx_2dx_3 + dx_3^2$

group:

$$X_1 f = \frac{\partial f}{\partial x_2}\ ,\ X_2 f = \frac{\partial f}{\partial x_3}\ ,\ X_3 f = \frac{\partial f}{\partial x_1} - x_2\frac{\partial f}{\partial x_2}\ ,$$
$$X_4 f = x_2\frac{\partial f}{\partial x_1} + \frac{1}{2}\left(\frac{e^{-2x_1}}{1-n^2} - x_2^2\right)\frac{\partial f}{\partial x_2} - \frac{ne^{-x_1}}{1-n^2}\frac{\partial f}{\partial x_3}$$

composition:

$$[X_1, X_2] = 0\ ,\ [X_1, X_3]f = -X_1 f\ ,\ [X_1, X_4]f = X_3 f\ ,$$
$$[X_2, X_3] = 0\ ,\ [X_2, X_4] = 0\ ,\ [X_3, X_4]f = -X_4 f$$

c) *[Type IX]*

$ds^2 = dx_1^2 + (\sin^2 x_1 + n^2\cos^2 x_1)\,dx_2^2 + 2n\cos x_1\,dx_2dx_3 + dx_3^2$

group:

$$X_1 f = \frac{\partial f}{\partial x_2}\ ,\ X_2 f = \cos x_2\frac{\partial f}{\partial x_1} - \cot x_1 \sin x_2\frac{\partial f}{\partial x_2} + \frac{n\sin x_2}{\sin x_1}\frac{\partial f}{\partial x_3}\ ,$$
$$X_3 f = -\sin x_2\frac{\partial f}{\partial x_1} - \cot x_1\cos x_2\frac{\partial f}{\partial x_2} + \frac{n\cos x_2}{\sin x_1}\frac{\partial f}{\partial x_3}\ ,\ X_4 f = \frac{\partial f}{\partial x_3}$$

composition:

$$[X_1, X_2]f = X_3 f\ ,\ [X_2, X_3]f = X_1 f\ ,\ [X_3, X_1]f = X_2 f\ ,$$
$$[X_1, X_4] = [X_2, X_4] = [X_3, X_4] = 0$$

Category F

Groups G_6 — spaces of constant curvature

39 Conclusion.

Having classified all possible types of spaces which admit a continuous group of motions, it remains only that we say how, given the line element of a space, one can verify whether that same space admits a continuous group of motions, and if so, how the equations are found which reduce the line element to one of the typical forms of our table.

For this purpose it is enough to recall the equations (A) §1 which are precisely according to Lie, the equations of definition[85] of the group. With only algebraic operations and differentiation one evaluate the number r of parameters of the group and decides on its transitivity or intransitivity,[86] so that one sees immediately to which of our categories the given space belongs.

The integration of the fundamental equations (A) then gives us the actual form of the infinitesimal transformations of the group and this makes the composition evident to us, after which one will decide immediately to which type in the category the space belongs since one will clearly find in the table one and only one group which offers the same or an equivalent composition.

Then one tries to identify the two groups, namely to assign the values of the constants which enter in the group of canonical form and to calculate the equations of transformation. To this task one responds perfectly applying the general criteria for the similarity of groups given in the work of Lie.

NOTE

After the editing of the present *Memoria* Professor Ricci brought my attention to a *Nota* of Professor Levi-Civita, where by chance particular 3-dimensional spaces with 3 or 4-parameter groups of motions are already given (see T. Levi-Civita, *Sul moto di un corpo rigido attorno ad un punto fisso* [On the motion of a rigid body around a fixed point], Rendiconti della Reale Accademia dei Lincei (5), 5 (2nd sem. 1896), 3–9; 122–123).

[87]

[85] S. Lie-F. Engel, Vol 1, §50, p.186.

[86] S. Lie-F. Engel, Vol 1, p.217.

[87] The original paper has a "correzione" here that corrects a sentence at the end of sec. 21. In the translation, the appropriate correction was made where it belongs [Editor].