The Bianchi Classification of 3-Dimensional Lie Algebras: Lezioni sulla teoria dei gruppi continui finite di trasformazioni (1918) §198–199 (pp. 550-557)

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§198. The Seven Types of Compositions of Integrable G_3's

The direction of our investigation leads us to determine all the spaces S_3 which admit a *transitive* and therefore simply transitive group G_3 of motions. And since two such G_3's are always similar when equally composed (§98), the first classification to make is that of all possible compositions for a G_3; then having chosen a specific representative for each of these types, we have to study the spaces which admit the G_3 as a (complete or partial) group of motions. In this and the following section we occupy ourselves with the preliminary study of the types of compositions for a G_3.

We distinguish the G_3's according to whether they are integrable or nonintegrable. If a G_3 is integrable, its derived group contains less than 3 parameters (§81 *a*)), and conversely if this happens the G_3 is integrable since the derived group, having fewer than 3 parameters, is certainly integrable. A G_3 is therefore not integrable only when it coincides with its own derived group, in which case it is simple.[1]

We treat in this section the case of an integrable G_3 . Its derived group will fall into one of the following three categories:

a) it reduces to the identity

b) it is a $G_1 \equiv [X_1 f]$

c) it is a $G_2 \equiv [X_1 f, X_2 f]$.

[1]It cannot possess an invariant G_2 (§81 *a*)), nor even an invariant G_1 since in such a case the derived group would have at most two parameters.

Case $a)$

The group G_3 is abelian and offers the first and simplest composition

Type I : $$[X_1, X_2]f = [X_1, X_3]f = [X_2, X_3]f = 0 \ .$$

Case $b)$

Here we have

$$[X_1, X_2]f = \alpha X_1 f \ , \ [X_1, X_3]f = \beta X_1 f \ , \ [X_2, X_3]f = \gamma X_1 f \ .$$

with α, β, γ constants which are not simultaneously zero (otherwise we would be in case a)). But one of the three, for instance α, can be made zero; it suffices (when $\alpha \neq 0$) to change $X_2 f$ into $\bar{X}_2 f = -\beta/\alpha \, X_2 f + X_3 f$. Therefore we can assume

$$[X_1, X_2]f = 0 \ , \ [X_1, X_3]f = \beta X_1 f \ , \ [X_2, X_3]f = \gamma X_1 f \ .$$

and if also $\beta = 0$ it suffices to multiply $X_3 f$ by a constant factor to make $\gamma = 1$ leading to

Type II : $$[X_1, X_2]f = [X_1, X_3]f = 0 \ , \ [X_2, X_3]f = X_1 f \ .$$

If then $\beta \neq 0$ we can make $\beta = 1$ and render $\gamma = 0$, if it is not already so, by changing $X_2 f$ into $\bar{X}_1 f - 1/\gamma \, X_2 f$ so that

$$[X_1, X_2]f = [X_1, X_3]f - 1/\gamma \, [X_2, X_3]f = 0 \ .$$

Therefore we obtain[2]

Type III : $$[X_1, X_2]f = 0 \ , \ [X_1, X_3]f = X_1 f \ , \ [X_2, X_3]f = 0 \ .$$

Case $c)$

We begin by proving that in this case the derived group $[X_1 f, X_2 f]$ is necessarily abelian. Suppose on the contrary that $[X_1, X_2]f = aX_1 f + bX_2 f$ with the constants a, b *not both zero*, and one has

$$[X_1, X_3]f = \alpha X_1 f + \beta X_2 f \ , \ [X_2, X_3]f = \gamma X_1 f + \delta X_2 f \ .$$

The Jacobi identity $[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0$ gives among the constants the two relations $b\gamma - a\delta = 0$, $b\alpha - a\beta = 0$, so that, a and b not both zero by hypothesis, we must have $\alpha\delta - \beta\gamma = 0$. The matrix

[2] That the types II, III are actually different is shown for example by the observation that the derived group $[X_1 f]f$ is contained in the first case in ∞^1 abelian G_2's $[X_1 f, aX_2 f + bX_3 f]$ but in the second only in the abelian G_2 $[X_1 f, X_2 f]$.

$\begin{pmatrix} a & \alpha & \gamma \\ b & \beta & \delta \end{pmatrix}$ would therefore have rank 1 and the three linear forms in $X_1 f$, $X_2 f$,

$$aX_1 + bX_2 \ , \ \alpha X_1 + \beta X_2 \ , \ \gamma X_1 + \delta X_2$$

would be reducible to only one, in other words the derived group would fall into the preceding case b).

The present composition will therefore be

$$\text{(A)} \ [X_1, X_2]f = 0 \ , \ [X_1, X_3]f = \alpha X_1 f + \beta X_2 f \ , \ [X_2, X_3]f = \gamma X_1 f + \delta X_2 f \ ,$$

$$\alpha\delta - \beta\gamma \neq 0 \ .$$

We try to see if by changing $X_1 f$ into $\bar{X}_1 f = aX_1 f + bX_2 f$, one can make $\beta = 0$, namely $[\bar{X}_1, X_3]f = \rho \bar{X}_1 f$ (ρ constant).

For this we must have

$$a(\alpha X_1 f + \beta X_2 f) + b(\gamma X_1 f + \delta X_2 f) = \rho \bar{X}_1 f = \rho(aX_1 f + bX_2 f)$$

which reduces to

$$a(\alpha - \rho) + b\gamma = 0 \ , \ a\beta + b(\gamma - \rho) = 0 \ . \tag{34}$$

It therefore suffices for ρ to satisfy the quadratic equation

$$\rho^2 - (\alpha + \delta)\rho + \alpha\delta - \beta\gamma = 0 \ . \tag{35}$$

and from (34) a, b (the ratio a/b) can be calculated. But, from our real point of view, it is necessary to distinguish the cases where (35) has real or imaginary roots. Assuming first that they are real, we are reduced to the composition

$$[X_1, X_2]f = 0 \ , \ [X_1, X_3]f = \rho X_1 f \ , \ [X_2, X_3]f = \gamma X_1 f + \delta X_2 f \ ,$$

where since $\rho \neq 0$, we can make $\rho = 1$ and have

$$[X_1, X_2]f = 0 \ , \ [X_1, X_3]f = X_1 f \ , \ [X_2, X_3]f = \gamma X_1 f + \delta X_2 f \ . \tag{36}$$

If δ which is certainly not zero, is $= 1$, then when $\gamma \neq 0$, by changing $X_1 f$ into $1/\gamma \, X_1 f$, we can also make $\gamma = 1$ and have the composition

Type IV : $\quad [X_1, X_2]f = 0 \ , \ [X_1, X_3]f = X_1 f \ , \ [X_2, X_3]f = X_1 f + X_2 f \ .$

and in the case where $\gamma = 0$ the other one

Type V : $\quad [X_1, X_2]f = 0 \ , \ [X_1, X_3]f = X_1 f \ , \ [X_2, X_3]f = X_2 f \ .$

When $\delta \neq 1$, by changing $X_2 f$ into $\bar{X}_2 f = X_2 f + (\gamma/(\delta - 1) \, X_1 f$, keeping the same first two equations of the composition (36), one makes $\gamma = 0$ and has the new type of composition
Type VI :

$$[X_1, X_2]f = 0 \ , \ [X_1, X_3]f = X_1 f \ , \ [X_2, X_3]f = hX_2 f \ , \quad (h \neq 0, 1) \ .$$

Finally it remains for us to return to the general equations (A) to consider the case in which the quadratic equation (35) has complex roots, so that namely

$$(\alpha - \delta)^2 + 4\beta\gamma < 0 \ . \tag{37}$$

In this case, if α is not already zero in (A), we can make it so by changing $X_1 f$ into $\bar{X}_1 f = X_2 f - \gamma/\alpha\, X_1 f$ so that then absorbing the factor β into $X_2 f$,[3] we have

$$[X_1, X_2]f = 0 \ , \ [X_1, X_3]f = X_2 f \ , \ [X_2, X_3]f = \gamma X_1 f + \delta X_2 f \ ,$$

and then by (37)

$$\delta^2 + 4\gamma < 0 \ . \tag{38}$$

Now letting

$$\bar{X}_1 f = a X_1 f \ , \ \bar{X}_2 f = b X_2 f \ , \ \bar{X}_3 f = c X_3 f \ ,$$

we have

$$[\bar{X}_1, \bar{X}_2]f = 0 \ , \ [\bar{X}_1, \bar{X}_3]f = ac/b\, \bar{X}_2 f \ , \ [\bar{X}_2, \bar{X}_3]f = bc(\gamma/\alpha\, \bar{X}_1 f + \delta/b\, \bar{X}_2 f) \ .$$

We choose the constants a, b, c in such a way that one has $ac = b$, $bc\gamma = -a$, which can be done in a real manner, since by (38) γ is negative; it suffices to take for example: $a = 1, b = c = \sqrt{-1/\gamma}$.

Therefore we have the last type of composition for the integrable G_3's

$$\text{Type VII}: \quad [X_1, X_2]f = 0 \ , \ [X_1, X_3]f = X_2 f \ , \ [X_2, X_3]f = -X_1 f + h X_2 f \ ,$$

$$\text{with } (h^2 < 4) \text{ by (38)} \ ,$$

(since $h = c\delta = \delta\sqrt{-\gamma}$, $h^2/4 = \delta^2/(-4\gamma) < 1$).

This type differs from the preceding six because it does not contain any real invariant one parameter subgroup, while the others contain at least one.

§199. The Two Types of Nonintegrable (simple) G_3's

For a nonintegrable G_3 the derived group coincides with the G_3 so that in the corresponding equations of composition

$$\begin{aligned} [X_2, X_3]f &= C^1{}_{23} X_1 f + C^2{}_{23} X_2 f + C^3{}_{23} X_3 f \ , \\ [X_3, X_1]f &= C^1{}_{31} X_1 f + C^2{}_{31} X_2 f + C^3{}_{31} X_3 f \ , \\ [X_1, X_2]f &= C^1{}_{12} X_1 f + C^2{}_{12} X_2 f + C^3{}_{12} X_3 f \ , \end{aligned} \tag{39}$$

[3]Translator's note:

$[\bar{X}_1, \bar{X}_3]f = [X_2, X_3]f - \gamma/\alpha\, [X_1, X_3]f = \gamma X_1 + \delta X_2 - \gamma/\alpha\, (\alpha X_1 + \beta X_2) = (\delta - \beta\gamma/\alpha) X_2$

but $\alpha\delta - \beta\gamma \neq 0$ from (A) so we can make $\delta - \beta\gamma/\alpha = 1$ i.e., "absorb the factor β into $X_2 f$."

the determinant $|C| = \begin{vmatrix} C^1{}_{23} & C^2{}_{23} & C^3{}_{23} \\ C^1{}_{31} & C^2{}_{31} & C^3{}_{31} \\ C^1{}_{12} & C^2{}_{12} & C^3{}_{12} \end{vmatrix}$ *is not zero.*

Furthermore we can see that the determinant C is *symmetric*, namely

$$C^1{}_{31} = C^2{}_{23} \ , \ C^1{}_{12} = C^3{}_{23} \ , \ C^2{}_{12} = C^3{}_{31} \ . \qquad (40)$$

This follows from the Jacobi identity

$$[[X_2, X_3], X_1]f + [[X_3, X_1], X_2]f + [[X_1, X_2], X_3]f = 0 \ ,$$

which by (39) can be written

$$\begin{aligned} &C^2{}_{23}[X_2, X_1] + C^3{}_{23}[X_3, X_1] + C^1{}_{31}[X_1, X_2] \\ &\quad + C^3{}_{31}[X_3, X_2] + C^1{}_{12}[X_1, X_3] + C^2{}_{12}[X_2, X_3] = 0 \ , \end{aligned}$$

so that

$$(C^1{}_{31} - C^2{}_{23})[X_1, X_2] + (C^2{}_{12} - C^3{}_{31})[X_2, X_3] + (C^3{}_{23} - C^1{}_{12})[X_3, X_1] = 0 \ ,$$

and since the three commutators are linearly independent, (40) follows from it.

Now we take any two infinitesimal transformations Xf, Yf of G_3; let

$$Xf = x_1X_1f + x_2X_2f + x_3X_3f \ , \quad Yf = y_1X_1f + y_2X_2f + y_3X_3f \ ,$$

where the coefficients, indicated by (x_1, x_2, x_3), (y_1, y_2, y_3) will be interpreted as the homogeneous coordinates of points in a plane, so that every infinitesimal transformation of the G_3 has a point of the plane as an image, from which it is inversely determined in an unequivocal way. We examine how the coefficients (z_i) of the commutator

$$Zf = [X, Y]f = z_1X_1f + z_2X_2f + z_3X_3f$$

depend on those (x_i), (y_i) of the factors. One has

$$[X, Y]f = (x_2y_3 - x_3y_2)[X_2, X_3]f + (x_3y_1 - x_1y_3)[X_3, X_1]f + (x_1y_2 - x_2y_1)[X_1, X_2]f$$

and the three binomials

$$\xi_1 = x_2y_3 - x_3y_2 \ , \ \xi_2 = x_3y_1 - x_1y_3 \ , \ \xi_3 = x_1y_2 - x_2y_1$$

are precisely the coordinates of the line which joins the two points (x_i), (y_i). From equations (39) one obtains

$$\begin{aligned} z_1 &= C^1{}_{23}\xi_1 + C^1{}_{31}\xi_2 + C^1{}_{12}\xi_3 \ , \\ z_2 &= C^2{}_{23}\xi_1 + C^2{}_{31}\xi_2 + C^2{}_{12}\xi_3 \ , \\ z_3 &= C^3{}_{23}\xi_1 + C^3{}_{31}\xi_2 + C^3{}_{12}\xi_3 \ , \end{aligned} \qquad (41)$$

and these, since the determinant C is nonzero and symmetric, are the equations of an invertible non-degenerate correlation between the point (z_i) of the plane and the line (ξ_i). This correlation is therefore a polarity with respect to a certain conic Γ in the plane, from which by (41) we immediately write its equation in the line coordinates ξ

$$\begin{aligned} C^1{}_{23}\xi_1{}^2 + C^2{}_{31}\xi_2{}^2 + C^3{}_{12}\xi_3{}^2 + (C^1{}_{31} + C^2{}_{23})\xi_1\xi_2 \\ +(C^2{}_{12} + C^3{}_{31})\xi_2\xi_3 + (C^3{}_{23} + C^1{}_{12})\xi_3\xi_1 = 0 \end{aligned} \tag{42}$$

Therefore: *The image point of the commutator* $[X, Y]$ *is the pole, with respect to the fundamental conic* Γ *with the tangential equation (42), of the line joining the two image points of the factors* Xf, Yf.

Now if the generating transformations X_1, X_2, X_3 are replaced by three linearly independent combinations of themselves, this is equivalent to performing a transformation of the coordinates (x_i). We can take advantage of this to reduce the equation of the conic Γ, and consequently the equations of composition, to a determined canonical form. The only distinction to be made is whether or not the conic Γ whose equation has real coefficients, is real or complex.

Case 1. Γ is a real conic. Taking the triangle of reference formed by the two tangents to the conic and the cord joining the points of contact, we will give the equation (42) of Γ the canonical form $\xi_1\xi_2 - \xi_2{}^2 = 0$ so that $C^3{}_{23} = C^1{}_{12} = 1$, $C^2{}_{31} = -2$, all the other C's are zeros, and we have the type of composition

Type VIII : $\quad [X_1, X_2]f = X_1 f\ ,\ [X_1, X_3]f = 2X_2 f\ ,\ [X_2, X_3]f = X_3 f\ .$

Case 2. If the conic is complex, one takes a triangle of reference self-conjugate with respect to Γ and gives equation (42) the form

$$\xi_1{}^2 + \xi_2{}^2 + \xi_3{}^2 = 0\ .$$

Therefore $C^1{}_{23} = C^2{}_{31} = C^3{}_{12} = 1$ and all the other C's are zero, so that we have as the last type

Type IX : $\quad [X_1, X_2]f = X_3 f\ ,\ [X_2, X_3]f = X_1 f\ ,\ [X_3, X_1]f = X_2 f\ .$

The composition VIII is that of the group of motions of the non-Euclidean plane (hyperbolic geometry), the other IX that of the group of motions of the sphere into itself (elliptical geometry) and they differ in that the type VIII admits real 2-parameter subgroups, which do not exist for the type IX (see §186).